

Local Wick Polynomials and Time Ordered Products of Quantum Fields in Curved Spacetime

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Abstract

In order to have well defined rules for the perturbative calculation of quantities of interest in an interacting quantum field theory in curved spacetime, it is necessary to construct Wick polynomials and their time ordered products for the noninteracting theory. A construction of these quantities has recently been given by Brunetti, Fredenhagen, and Köhler, and by Brunetti and Fredenhagen, but they did not impose any “locality” or “covariance” condition in their constructions. As a consequence, their construction of time ordered products contained ambiguities involving arbitrary functions of spacetime point rather than arbitrary parameters. In this paper, we construct an “extended Wick polynomial algebra”—large enough to contain the Wick polynomials and their time ordered products—by generalizing a construction of Dütsch and Fredenhagen to curved spacetime. We then define the notion of a *local, covariant quantum field*, and seek a definition of *local* Wick polynomials and their time ordered products as local, covariant quantum fields. We introduce a new notion of the scaling behavior of a local, covariant quantum field, and impose scaling requirements on our local Wick polynomials and their time ordered products as well as certain additional requirements—such as commutation relations with the free field and appropriate continuity properties under variations of the spacetime metric. For a given polynomial order in powers of the field, we prove that these conditions uniquely determine the local Wick polynomials and their time ordered products up to a finite number of parameters. (These parameters correspond to the usual renormalization ambiguities occurring in Minkowski spacetime together with

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additional parameters corresponding to the coupling of the field to curvature.) We also prove existence of local Wick polynomials. However, the issue of existence of local time ordered products is deferred to a future investigation.

1 Introduction

Despite some important differences from quantum field theory in Minkowski spacetime caused by the lack of a “preferred vacuum state”, the theory of a linear quantum field in a globally hyperbolic, curved spacetime is entirely well formulated (see, e.g., [14, 20] for a review). However, even in Minkowski spacetime, the theory of a nonlinear (i.e., self-interacting) quantum field is not, in general, well formulated. Nevertheless, in Minkowski spacetime there are well defined rules for obtaining perturbation series expressions for all quantities of interest for a nonlinear field (and in particular the interacting field itself). These perturbation expressions are defined up to certain, well specified “renormalization ambiguities”. It is of interest to know if a similar perturbative definition of nonlinear quantum fields can be given in curved spacetime and, if so, whether the renormalization ambiguities in curved spacetime are of the same nature as those in Minkowski spacetime.

This issue was analyzed by Bunch and collaborators [3, 4], but the key steps in this analysis were done in the context of Riemannian spaces rather than Lorentzian spacetimes. Now, Minkowski spacetime can be viewed as a real section of a complex 4-dimensional space that also contains a 4-dimensional, real Euclidean section. It is well known that a suitable definition of a field theory on this Euclidean section gives rise (via analytic continuation) to the definition of a field theory in Minkowski spacetime. However, no such connection between Riemannian and Lorentzian field theory holds for curved spacetimes, since (apart from a few special classes of spacetimes, such as static spacetimes) a general Lorentzian spacetime cannot be expressed as a section of a complex spacetime that also contains a real Riemannian section. Furthermore, the techniques used by Bunch cannot readily be generalized to the Lorentzian case because of the very significant mathematical differences in the nature of the divergences occurring in the Riemannian and Lorentzian cases. For example, in the Riemannian case, it follows from elliptic regularity that Green’s functions for the free theory are unique up to addition of smooth functions. However, no such result holds in the Lorentzian case, as exemplified by the very different properties of the advanced, retarded, and Feynman propagators. Furthermore, singularities in the Green’s function occur only in the coincidence limit in the Riemannian case, but they occur also for non-coincident, lightlike related events in the Lorentzian case. As a result, formulas like (2.14) of [3], which play a crucial role in the Riemannian analysis, cannot be readily taken over to the Lorentzian case. In addition, dimensional regularization and other renormalization techniques used in Riemannian spaces are not well defined in Lorentzian spacetimes.

Recently, significant progress in the definition of perturbative quantum field theory in Lorentzian spacetimes was made by Brunetti and Fredenhagen [1, 2], who used the methods of “microlocal analysis” [11, 7] to analyze the nature of the divergences occurring in the Lorentzian theory. In [1], these authors considered the Fock space arising (via the GNS construction) from a choice of quasi-free Hadamard state ω . They showed that on this Hilbert space, the Wick polynomials—generated by the (formally infinite) products

of field operators and their derivatives evaluated at the same spacetime point—can be given a well defined meaning as operator-valued-distributions via a normal ordering prescription with respect to ω . In [2], they then used an adaptation of the Epstein-Glaser method [8] of renormalization in Minkowski spacetime to analyze time ordered products of Wick polynomials, which are the quantities needed for a perturbative construction of the interacting field theory. They thereby showed that quantum field theories in curved spacetime could be given the same “perturbative classification” as in Minkowski spacetime, i.e., that all of the “ultraviolet divergences” of the theory in curved spacetime are of the same nature as in Minkowski spacetime. Nevertheless, their analysis in curved spacetime left open a much greater renormalization ambiguity than in Minkowski spacetime: In essence, quantities that appear at each perturbation order in Minkowski spacetime as renormalized coupling *constants* now appear in curved spacetime as renormalized coupling *functions*, whose dependence upon the spacetime point can be arbitrary.

It seems clear that the missing ingredient in the analysis of [2] is the imposition of a suitable requirement of covariance/locality on the renormalization prescription, as was previously given for the definition of the stress-energy tensor of a free quantum field (see pp. 89–91 of [20]). The imposition of such a condition should provide an appropriate replacement for the imposition of Poincare invariance in Minkowski spacetime. When such a condition is imposed, one would expect that the renormalized coupling functions would no longer be arbitrary functions of the spacetime point but would be locally constructed out of the metric in a covariant manner. Furthermore, one might expect that when suitable continuity and scaling requirements are also imposed, the ambiguities should be reduced to finitely many free parameters at each order rather than free functions. The renormalization ambiguities would then correspond to the renormalization ambiguities in Minkowski spacetime together with the renormalization of some additional parameters associated with couplings of the quantum field to curvature.

The main purpose of this paper is to show that these expectations are correct with regard to the uniqueness (though not necessarily the existence) of the perturbatively defined theory. A key step in our analysis is to define the notion of a *local¹, covariant quantum field*. The basic idea behind this notion is to consider a situation wherein one changes the metric outside of some region O and, in essence, demands that the local, covariant quantum field not change within O . A precise definition of this notion will be given in Section 3 below (see Def. 3.2). In Section 3, we will also explicitly see that the Wick polynomials as defined in [1] fail to be local, covariant quantum fields (no matter how ω is chosen); consequently, neither are the time ordered products of these fields constructed in [2]. These quantities must therefore not be used for the definition of the local observables in the interacting theory; their definition depends on a choice of a

¹In quantum field theory, the terminology “local field” is commonly used to mean a field that commutes with itself at spacelike separated events. Our use of the terminology “local, covariant field” here is not related to this notion. Rather, we use this terminology to express the idea that the field is constructed in a local and covariant way from the spacetime metric, as precisely defined in Section 3 below.

reference state ω , which is itself a highly nonlocal quantity.

Our analysis will proceed as follows: First, we will obtain, for any given globally hyperbolic spacetime (M, \mathbf{g}) , an abstract “extended Wick polynomial algebra”, $\mathcal{W}(M, \mathbf{g})$, via a normal ordering prescription with respect to a quasi-free Hadamard state, ω . (We refer to our algebra $\mathcal{W}(M, \mathbf{g})$ as “extended”, because it is actually enlarged beyond the usual Wick polynomial algebra so as to already include elements corresponding to the time ordered products of Wick polynomials.) Our construction of this algebra is essentially a straightforward generalization to curved spacetime (using the methods of [1]) of a construction previously given by [6] in the context of Minkowski space. We then note that the resulting operator algebra—viewed as an abstract algebra—is independent of the choice of ω .

Next, we will seek to identify the elements of this abstract algebra that merit the interpretation of representing the various Wick polynomials and time ordered products. As indicated above, the crucial requirement that we shall place on these elements is that they be local, covariant quantum fields. We shall refer to these elements as “*local* Wick polynomials” and “*local* time ordered products”. Some other “specific properties”—such as commutation relations with the free field—will also be imposed as requirements on the definitions of these quantities. It is worth emphasizing that, unlike in Minkowski space, we will find that some ambiguities necessarily arise in defining local Wick polynomials. Consequently, renormalization ambiguities in defining perturbative quantum field theory in curved spacetime arise not only from the definition of time ordered products of Wick polynomials but also from the definition of the local Wick polynomials themselves.

As indicated above, after our locality/covariance requirement and our other specific properties have been imposed on the definition of Wick polynomials and their time ordered products, we will find that the ambiguities in the definitions of these quantities will be reduced from arbitrary functions of the spacetime point to functions that are locally constructed from the metric (as well as parameters that appear in the classical theory) in a covariant manner. However, in order to further reduce the ambiguities to the renormalization of finitely many parameters at each order, there are two other conditions we must impose: (i) a suitable continuous/analytic dependence of the local Wick polynomials and their time ordered products on the metric, \mathbf{g} , and coupling constants, p , and (ii) a suitable scaling behavior of these quantities. However, neither of these notions are straightforward to define.

The difficulty with defining a suitable notion of the continuous dependence of an element in $\mathcal{W}(M, \mathbf{g})$ on the metric and parameters occurring in the classical theory arises from the fact that the Wick polynomial algebra $\mathcal{W}(M, \mathbf{g})$ for a spacetime (M, \mathbf{g}) is not naturally isomorphic to the Wick polynomial algebra $\mathcal{W}(M, \mathbf{g}')$ for a different spacetime (M, \mathbf{g}') , so it is far from clear what it means for an element of the Wick polynomial algebra to vary continuously as \mathbf{g} is continuously varied to \mathbf{g}' . Fortunately, the task of defining this notion is made much easier by the fact that we are concerned only with local, covariant quantum fields, so we may restrict attention to metric variations that

occur in some spacetime region O with compact closure. In order to make use of a similar simplification with regard to variations of the parameters, p , appearing in the classical theory, it is convenient to allow these parameters to become functions of spacetime point and to then also restrict attention to variations that occur only within O . If \mathbf{g} agrees with \mathbf{g}' and p agrees with p' outside of O , we can identify an element of $\mathcal{W}_p(M, \mathbf{g})$ with the element of $\mathcal{W}_{p'}(M, \mathbf{g}')$ which, say, agrees with it outside of future² of O (where we have put a subscript p on the algebras to indicate their dependence on the coupling parameters). With this identification of elements of the different algebras, we require that if $(\mathbf{g}^{(s)}, p^{(s)})$ vary smoothly with s in a suitable sense, then within O each local Wick polynomial and time ordered product of local Wick polynomials must vary continuously with s . A precise formulation of this requirement will be given in Section 4.2 below.

The above requirement that the local Wick polynomials and their time ordered products depend continuously on the metric would not suffice to eliminate non-analytic local curvature ambiguities of the sort considered in [19]. We therefore shall impose an additional analyticity requirement that states that if $\mathbf{g}^{(s)}$ is a one-parameter analytic family of analytic metrics, then each local Wick polynomial and time ordered product of local Wick polynomials must vary analytically with s ; we similarly require analytic variation of local Wick polynomials and their time ordered products under analytic variation of the parameters p . However, for analytic spacetimes, we cannot use the above method to identify algebras of different spacetimes, since one can no longer make local variations of the metric. Instead, we proceed by introducing a notion of an analytic family, $\omega^{(s)}$, of quasi-free Hadamard states on $(M, \mathbf{g}^{(s)})$, and we require that the distributions obtained by acting with $\omega^{(s)}$ on the local Wick polynomials and their time ordered products vary analytically with s in a suitable sense. A precise formulation of these requirements will be given in Section 4.2.

In Minkowski spacetime, scaling behavior is usually formulated in terms of how fields behave under the transformation $x \rightarrow \lambda x$. Such a formulation would be highly coordinate dependent in curved spacetime and thus would be very awkward to implement. Our notion of local, covariant quantum fields allows us to formulate a notion of scaling in terms of the behavior of these fields under the scaling of the spacetime metric, $\mathbf{g} \rightarrow \lambda^2 \mathbf{g}$ (where λ is a constant) together with associated scalings of the parameters, p , occurring in the theory. Note that in Minkowski spacetime, consideration of the behavior of a local, covariant quantum field under scaling of the spacetime metric, $\mathbf{g} \rightarrow \lambda^2 \mathbf{g}$, is equivalent to considering the behavior of these fields under $x \rightarrow \lambda x$, since this diffeomorphism is a conformal isometry with constant conformal factor λ^2 , so $x \rightarrow \lambda x$ with fixed metric is equivalent via a diffeomorphism to $\mathbf{g} \rightarrow \lambda^2 \mathbf{g}$ at each fixed x . If we consider a classical field theory that is invariant under $\mathbf{g} \rightarrow \lambda^2 \mathbf{g}$ together with corresponding scaling transformations on the field and on the parameters, $p \rightarrow p(\lambda)$, appearing in the theory,

²We would obtain a different identification of the algebras by demanding agreement outside the past of O , but this would give rise to an equivalent notion of continuous dependence.

then the corresponding field algebras, $\mathcal{W}_{p(\lambda)}(M, \lambda^2 \mathbf{g})$, will be naturally isomorphic to each other. It might appear natural to require that our definition of local Wick polynomials and their time ordered products be such that they are preserved under this isomorphism of the algebras. However, even in quantum field theory in Minkowski spacetime, it is well known that such a requirement cannot be imposed on time ordered products. In curved spacetime, we shall show that such a scaling requirement cannot be imposed upon the local Wick polynomials either. However, it is possible to require that the failure of the local Wick polynomials and their time ordered products to scale like their classical counterparts is given by terms with only logarithmic dependence upon λ . This notion is made precise in Section 4.3.

The main results of this paper may now be summarized. First, we shall construct the algebra $\mathcal{W}(M, \mathbf{g})$ for an arbitrary globally hyperbolic spacetime. We then define the notion of a “local, covariant quantum field” and provide an axiomatic characterization of “local Wick polynomials” and their time ordered products. We shall then prove the existence of local Wick polynomials via an explicit construction, and we shall give a precise characterization of their non-uniqueness. Next, we consider the time ordered products of local Wick polynomials. We shall obtain a precise characterization of the non-uniqueness of these time ordered products in a manner similar to our analysis of the non-uniqueness of the local Wick polynomials. However, the existence of time ordered products that satisfy our covariance/locality requirement cannot be readily proven because the Epstein-Glaser prescription does not manifestly preserve covariance/locality. Consequently, we shall defer the investigation of existence of time ordered products to a future investigation.

For simplicity and definiteness, we shall restrict consideration in this paper to the theory of a real scalar field. However, the generalization of our definitions and conclusions to other fields should be straightforward.

Notations and conventions: Throughout, (M, \mathbf{g}) denotes a globally hyperbolic, time-oriented spacetime. The manifold structure of M is assumed to be real analytic, and the metric tensor $\mathbf{g} \equiv g_{ab}$ is assumed to be smooth (but not necessarily analytic). Our conventions regarding the spacetime geometry are those of [21]. V_x^\pm denote the closed future resp. past lightcone at a point x . $\square_{\mathbf{g}} = g^{ab} \nabla_a \nabla_b$ is the wave operator in curved space and $\mu_{\mathbf{g}} = |\det \mathbf{g}|^{1/2} d^4x$. $\mathcal{D}(M)$ is the space of (complex-valued) test functions with compact support on M and $\mathcal{D}'(M)$ is the corresponding dual space of distributions. Our convention for the Fourier transform in \mathbb{R}^n is $\hat{u}(k) = (2\pi)^{-n/2} \int e^{+ikx} u(x) d^n x$.

2 Definition of the extended Wick-polynomial algebra

2.1 Definition of the fundamental algebra of observables associated with a quantized Klein-Gordon field

The theory of a free classical Klein-Gordon field on a spacetime (M, \mathbf{g}) with mass m and curvature coupling ξ is described by the action

$$\mathcal{S} = \int_M \mathcal{L}_0 \mu_{\mathbf{g}} = \int_M (g^{ab} \nabla_a \varphi \nabla_b \varphi + \xi R \varphi^2 + m^2 \varphi^2) \mu_{\mathbf{g}}. \quad (1)$$

The theory of a free quantized Klein-Gordon field in curved spacetime can be formulated in various ways. For our purposes, it is essential to formulate the theory within the so-called “algebraic approach” (see, for example [14, 20]). In this approach, one starts from an abstract $*$ -algebra $\mathcal{A}(M, \mathbf{g})$ (with unit), which is generated by certain expressions in the smeared quantum field, $\varphi(f)$, where f is a test function. In [14, 20], expressions of the form $e^{i\varphi(f)}$ were considered. The main advantage of working with such expressions is that the so-obtained algebra then has a norm (in technical terms, it is a C^* -algebra). Defining the algebra $\mathcal{A}(M, \mathbf{g})$ in that way would however be inconvenient for our purposes. Instead, we shall take $\mathcal{A}(M, \mathbf{g})$ to be the $*$ -algebra generated by the identity and the smeared field operators $\varphi(f)$ themselves, subject to the following relations:

Linearity: $\mathcal{D}(M) \ni f \rightarrow \varphi(f) \in \mathcal{A}(M, \mathbf{g})$ is complex linear.

Klein-Gordon: $\varphi((\square_{\mathbf{g}} - \xi R_{\mathbf{g}} - m^2)f) = 0$ for all $f \in \mathcal{D}(M)$.

Hermiticity: $\varphi(f)^* = \varphi(\bar{f})$.

Commutation Relations: $[\varphi(f_1), \varphi(f_2)] = i\Delta_{\mathbf{g}}(f_1 \otimes f_2)1$, where $\Delta_{\mathbf{g}} = \Delta_{\mathbf{g}}^{\text{adv}} - \Delta_{\mathbf{g}}^{\text{ret}}$ is the causal propagator for the Klein-Gordon operator.

The so-obtained algebra $\mathcal{A}(M, \mathbf{g})$ is now no longer a C^* -algebra, because of the unbounded nature of the smeared quantum fields $\varphi(f)$. This will however not be relevant in the following.

A state in the algebraic framework is a linear functional $\omega : \mathcal{A}(M, \mathbf{g}) \rightarrow \mathbb{C}$ which is normalized so that $\omega(1) = 1$ and positive in the sense that $\omega(a^*a) \geq 0$ for all $a \in \mathcal{A}(M, \mathbf{g})$. The algebraic notion of a state is related to the usual Hilbert-space notion of a state by the GNS theorem. This says that for any algebraic state ω , one can construct a Hilbert space \mathcal{H}_{ω} containing a distinguished “vacuum” vector $|\Omega_{\omega}\rangle$, and a representation π_{ω} of the algebraic elements $a \in \mathcal{A}(M, \mathbf{g})$ as linear operators on a dense invariant subspace $D_{\omega} \subset \mathcal{H}_{\omega}$, such that $\omega(a) = \langle \Omega_{\omega} | \pi_{\omega}(a) | \Omega_{\omega} \rangle$ for all $a \in \mathcal{A}(M, \mathbf{g})$. The multilinear functionals on $\mathcal{D}(M)$ defined by

$$\omega(f_1 \otimes \cdots \otimes f_n) \stackrel{\text{def}}{=} \omega(\varphi(f_1) \cdots \varphi(f_n)) \quad (2)$$

are called n -point functions. Every state on $\mathcal{A}(M, \mathbf{g})$ is uniquely determined by the collection of its n -point functions. A quasi-free state is by definition one which satisfies

$$\omega(e^{i\varphi(f)}) = e^{-\frac{1}{2}\omega(f \otimes f)}. \quad (3)$$

Note that the elements $e^{i\varphi(f)}$ do not actually belong to the algebra $\mathcal{A}(M, \mathbf{g})$. What is meant by Eq. (3) is the set of identities obtained by functionally differentiating this equation with respect to f . The so obtained identities then express the n -point functions of the state ω in terms of its two-point function. For quasi-free states, the GNS construction gives the usually considered representation of the fields on Fock-space, with $|\Omega_\omega\rangle$ the Fock-vacuum and with the field given in terms of creation and annihilation operators [14].

In our subsequent constructions, we will consider quasi-free states which are in addition of “global Hadamard type”. These are states whose two-point function has no spacelike singularities, and whose symmetrized two-point functions is given locally, modulo a smooth function, by a Hadamard fundamental solution [10], H , defined as

$$H(x, y) = u(x, y) P(\sigma^{-1}) + v(x, y) \ln |\sigma|. \quad (4)$$

Here, σ is the squared geodesic distance between the points x and y in the spacetime (M, \mathbf{g}) , u and v are certain real and symmetric smooth functions constructed from the metric and the couplings and “P” denotes the principal value. Strictly speaking, H is well defined only in analytic spacetimes (we will come back to this issue in Sec. 5.2), so the above definition needs to be modified in spacetimes that are only smooth. For a detailed discussion of this and of the statement that “there are no spacelike singularities”, see [14]. An immediate consequence the definition of Hadamard states is that if ω and ω' are Hadamard states, then $\omega(x, y) - \omega'(x, y)$ is a smooth function on $M \times M$.

There exists an alternative, equivalent characterization of globally Hadamard due to Radzikowski [18, Thm. 5.1], involving the notion of the “wave front set” [11, 7] of a distribution, which will play a crucial role in our subsequent constructions. (A definition of the wave front set and some of its elementary properties is given in the Appendix.) Namely, the globally Hadamard states in the sense of [14] are precisely those states whose two-point function is a bidistribution with wave front set

$$\text{WF}(\omega) = \{(x_1, k_1, x_2, -k_2) \in (T^*M)^2 \setminus \{0\} \mid (x_1, k_1) \sim (x_2, k_2), k_1 \in V_{x_1}^+\}. \quad (5)$$

Here, the following notation has been used: We write $(x_1, k_1) \sim (x_2, k_2)$ if x_1 and x_2 can be joined by a null geodesic and if k_1 and k_2 are cotangent and coparallel to that null geodesic.

2.2 Definition and properties of the algebra $\mathcal{W}(M, \mathbf{g})$

In the previous subsection, we reviewed the algebraic construction of a free quantum field theory. However, the algebra $\mathcal{A}(M, \mathbf{g})$ used in that construction includes only observables

corresponding to the smeared n -point functions of the free field. If we wish to define a nonlinear quantum field theory via a perturbative construction off the free field theory, we must consider additional observables, namely Wick polynomials and their time ordered products. Our strategy for doing so is to define an enlarged algebra of observables, $\mathcal{W}(M, \mathbf{g})$, that contains $\mathcal{A}(M, \mathbf{g})$ and also contains, among others, elements corresponding to (smeared) Wick polynomials of free-fields and (smeared) time ordered products of these fields. The construction of $\mathcal{W}(M, \mathbf{g})$ is essentially a straightforward generalization of [6], using ideas of [1, 2]. The construction initially depends on the choice of an arbitrary quasi-free Hadamard state ω on $\mathcal{A}(M, \mathbf{g})$. However, we will show below that different choices for ω give rise to isomorphic algebras. In that sense the algebras $\mathcal{W}(M, \mathbf{g})$ do not depend on the choice of a particular quasi-free Hadamard state. We note that, in particular, the construction of $\mathcal{W}(M, \mathbf{g})$ achieves the goal stated on p. 86 of [20], namely, to define an enlarged algebra of observables that includes the smeared stress-energy tensor.

Once we have properly identified the elements in $\mathcal{W}(M, \mathbf{g})$ corresponding to local Wick products and local time ordered products, the standard rules of perturbative quantum field theory will allow us to obtain perturbative expressions for the interacting field observables. These perturbative quantities—such as for example the interacting field itself—are given by formal power series in the coupling constants. The infinite sums occurring in these formal power series do not, of course, define elements of our algebra $\mathcal{W}(M, \mathbf{g})$. However, the expressions obtained by truncating these power series at some arbitrary order in perturbation theory will be elements in $\mathcal{W}(M, \mathbf{g})$. In that sense $\mathcal{W}(M, \mathbf{g})$ contains the observables (to arbitrary high order in perturbation theory) of the interacting theory. The “renormalization ambiguities” occurring in these perturbative expressions arise from the ambiguities in the definition of the local Wick products and local time ordered products. The main goal in this paper is to give a precise characterization of these ambiguities.

It should be noted that since $\mathcal{A}(M, \mathbf{g}) \subset \mathcal{W}(M, \mathbf{g})$, the notion of states for the nonlinear theory will be more restrictive than the notion of states for the free theory given in the previous section, but the states on $\mathcal{W}(M, \mathbf{g})$ will include a dense set of vectors in the GNS representation of any quasi-free Hadamard state. Indeed, it will follow from our results below that all Hadamard states on $\mathcal{A}(M, \mathbf{g})$ whose truncated n -point functions (other than the two-point function) are smooth can be extended to $\mathcal{W}(M, \mathbf{g})$. We conjecture that these are the only states on $\mathcal{A}(M, \mathbf{g})$ that can be extended to $\mathcal{W}(M, \mathbf{g})$, i.e., that the states on $\mathcal{W}(M, \mathbf{g})$ are in 1–1 correspondence with Hadamard states on $\mathcal{A}(M, \mathbf{g})$ with smooth truncated n -point functions.³

To begin our construction of $\mathcal{W}(M, \mathbf{g})$, choose a quasi-free Hadamard state ω on $\mathcal{A}(M, \mathbf{g})$. Via the GNS construction, one obtains from this a representation of the field

³ Kay (unpublished) has shown that in the vacuum representation of \mathcal{A} in Minkowski spacetime, these states include all n -particle states with smooth mode functions. More generally, he also showed that on a globally hyperbolic spacetime, these states include all n -particle states with smooth mode functions in the GNS representation of any quasi-free Hadamard state.

operators $\varphi(f)$ as linear operators on a Hilbert space \mathcal{H}_ω with dense, invariant domain D_ω , where we use the same symbol for the algebraic element $\varphi(f)$ and its representative on \mathcal{H}_ω . Next, define the symmetric operator-valued distributions

$$W_n(x_1, \dots, x_n) = :\varphi(x_1) \dots \varphi(x_n):_\omega \stackrel{\text{def}}{=} \frac{\delta^n}{i^n \delta f(x_1) \dots \delta f(x_n)} \exp \left[\frac{1}{2} \omega(f \otimes f) + i\varphi(f) \right] \Bigg|_{f=0} \quad (6)$$

for $n \geq 1$ and $W_0 \equiv 1$. The operators $W_n(t)$ obtained by smearing with a test function $t = f_1 \otimes \dots \otimes f_n \in \mathcal{D}(M^n)$ are elements the algebra $\mathcal{A}(M, \mathbf{g})$. The product of two operators $W_n(t)$ and $W_m(t')$ is given by the following formula (which is just a re-formulation of Wick's theorem),

$$W_n(t)W_m(t') = \sum_k W_{n+m-2k}(t \otimes_k t') \quad \forall t \in \mathcal{D}(M^n), t' \in \mathcal{D}(M^m). \quad (7)$$

The expression $t \otimes_k t'$ is the symmetrized, k times contracted tensor product, defined for $m, n \geq k$ by

$$(t \otimes_k t')(x_1, \dots, x_{n+m-2k}) \stackrel{\text{def}}{=} \mathbf{S} \frac{n!m!}{(n-k)!(m-k)!k!} \int_{M^{2k}} t(y_1, \dots, y_k, x_1, \dots, x_{n-k}) \times \\ t'(y_{k+1}, \dots, y_{k+i}, x_{n-k+1}, \dots, x_{n+m-2k}) \prod_{i=1}^k \omega(y_i, y_{k+i}) \mu_{\mathbf{g}}(y_i) \mu_{\mathbf{g}}(y_{k+i}) \quad (8)$$

where \mathbf{S} means symmetrization in x_1, \dots, x_{n+m-2k} . If either $m < k$ or $n < k$, then the contracted tensor product is defined to be zero.

In order to obtain more general operators such as normal ordered Wick powers, we would like to be able to smear the operator-valued distributions W_n not only with smooth test *functions*, but in addition also with certain compactly supported test *distributions* t . That this is indeed possible can be seen by means of a microlocal argument, which is based on the following observation [2]: The domain D_ω contains a dense invariant subspace of vectors $|\psi\rangle$ (the so-called “microlocal domain of smoothness”, see [2, Eq. (11)]) having the property that the wave front set of the vector-valued distributions $t \rightarrow W_n(t)|\psi\rangle$ is contained in the set $\mathbf{F}_n(M, \mathbf{g})$, defined as

$$\mathbf{F}_n(M, \mathbf{g}) = \{(x_1, k_1, \dots, x_n, k_n) \in (T^*M)^n \setminus \{0\} \mid k_i \in V_{x_i}^-, i = 1, \dots, n\}. \quad (9)$$

Now, smearing the above vector-valued distributions with a distributional test function t involves taking the pointwise product of two distributions. As it is well known, the pointwise product of two distributions is in general ill-defined. However, a theorem by

Hörmander [11, Thm. 8.2.10] states that if the wave front sets of two distributions u and v are such that $\{0\} \notin \text{WF}(u) + \text{WF}(v)$, then the pointwise product between u and v can be unambiguously defined. In the case at hand, we are thus allowed to smear W_n in with any compactly supported distribution t such that $\{0\} \notin \text{WF}(t) + \mathbf{F}_n(M, \mathbf{g})$. We here shall consider a subclass of the set of all such n -point distributions t , namely the class

$$\mathcal{E}'_n(M, \mathbf{g}) \stackrel{\text{def}}{=} \{t \in \mathcal{D}'(M^n) \mid t \text{ is symmetric, } \text{supp}(t) \text{ is compact, } \text{WF}(t) \subset \mathbf{G}_n(M, \mathbf{g})\}, \quad (10)$$

where

$$\mathbf{G}_n(M, \mathbf{g}) \stackrel{\text{def}}{=} (T^*M)^n \setminus \left(\bigcup_{x \in M} (V_x^+)^n \cup \bigcup_{x \in M} (V_x^-)^n \right). \quad (11)$$

Smearing W_n with test distributions $t \in \mathcal{E}'_n(M, \mathbf{g})$ gives therefore well defined operators on the microlocal domain of smoothness. (For notational simplicity, we denote this domain again by D_ω .)

Definition 2.1. $\mathcal{W}(M, \mathbf{g})$ is the $*$ -algebra of operators on \mathcal{H}_ω generated by 1 and elements of the form $W_n(t)$, where $n \geq 1$ and where $t \in \mathcal{E}'_n(M, \mathbf{g})$.

Theorem 2.1. The product in the algebra $\mathcal{W}(M, \mathbf{g})$ can be computed by Eq. (7), and the $*$ -operation is given by $W_n(t)^* = W_n(\bar{t})$. Furthermore, $W_n(t) = 0$ whenever t is of the form $t(x_1, \dots, x_n) = (\square_{\mathbf{g}} - \xi R_{\mathbf{g}} - m^2)_{x_i} s(x_1, \dots, x_n)$ for some $s \in \mathcal{E}'_n(M, \mathbf{g})$.

Proof. The statement concerning the $*$ -operation is obvious. In order to show that the algebra product can be calculated by Eq. (7), we first show that if $t \in \mathcal{E}'_n(M, \mathbf{g})$ and $t' \in \mathcal{E}'_m(M, \mathbf{g})$, then $t \otimes_k t' \in \mathcal{E}'_{n+m-2k}(M, \mathbf{g})$. Clearly $t \otimes_k t'$ is compactly supported and symmetric. We must show that in addition $\text{WF}(t \otimes_k t') \subset \mathbf{G}_{n+m-2k}(M, \mathbf{g})$. This can be seen by an application of [11, Thm. 8.2.13], which yields, in combination with Eq. (5) for $\text{WF}(\omega)$,

$$\begin{aligned} \text{WF}(t \otimes_k t') \subset & \{(x_1, k_1, \dots, x_{n+m-2k}, k_{n+m-2k}) \in (T^*M)^{n+m-2k} \mid \\ & \exists \text{ elements } (x_1, k_1, \dots, x_{n-k}, k_{n-k}, y_1, p_1, \dots, y_k, p_k) \in \text{WF}(t) \text{ and} \\ & (x_{n-k+1}, k_{n-k+1}, \dots, x_{n+m-2k}, k_{n+m-2k}, y_{k+1}, p_{k+1}, \dots, y_{2k}, p_{2k}) \in \text{WF}(t') \\ & \text{such that either } (x_j, p_j) \sim (x_{j+k}, -p_{j+k}) \text{ and } p_j \in V_{x_j}^- \setminus \{0\} \\ & \text{or } p_j = p_{j+k} = 0 \text{ for all } j = 1, \dots, k\}. \end{aligned} \quad (12)$$

It is not difficult to see that the set on the right side of the above inclusion is in fact contained in $\mathbf{G}_{n+m-2k}(M, \mathbf{g})$, thereby showing that $t \otimes_k t'$ is in the class $\mathcal{E}'_{n+m-2k}(M, \mathbf{g})$, as we wanted to show. We finish the proof by showing that Eq. (7) holds not only for smooth test functions, but also for our admissible test distributions $t \in \mathcal{E}'_n(M, \mathbf{g})$

and $t' \in \mathcal{E}'_m(M, \mathbf{g})$. To see this, we consider sequences of test functions $\{t_\alpha\}$ and $\{t'_\alpha\}$ converging to t and t' in the sense of $\mathcal{D}'_{\Gamma_n}(M^n)$ resp. $\mathcal{D}'_{\Gamma_m}(M^m)$ (for a definition of these spaces and their pseudo topology, the so-called ‘‘Hörmander pseudo topology’’, see the Appendix), where Γ_n and Γ_m are closed conic sets in $\mathbf{G}_n(M, \mathbf{g})$ and $\mathbf{G}_m(M, \mathbf{g})$, respectively with the property that $\text{WF}(t) \subset \Gamma_n$ and $\text{WF}(t') \subset \Gamma_m$. Now the operation of composing distributions—which forms the basis of the definition of the contracted tensor product, Eq. (8)—is continuous in the Hörmander pseudo topology. Therefore $t_\alpha \otimes_k t'_\alpha \rightarrow t \otimes_k t'$ in the space $\mathcal{D}'_{\Gamma_{n+m-2k}}(M^{n+m-2k})$, where Γ_{n+m-2k} is a certain closed conic set in $\mathbf{G}_{n+m-2k}(M, \mathbf{g})$, which is calculable from Γ_n and Γ_m using formula Eq. (12).

Now expressions of the sort $W_n(t)|\psi\rangle$ arise from the pointwise product of distributions. This product is continuous in the Hörmander pseudo topology. Therefore we conclude that $W_{n+m-2k}(t_\alpha \otimes_k t'_\alpha)|\psi\rangle \rightarrow W_{n+m-2k}(t \otimes_k t')|\psi\rangle$. By a similar argument, it also follows that $W_n(t_\alpha)W_m(t'_\alpha)|\psi\rangle \rightarrow W_n(t)W_m(t')|\psi\rangle$. Eq. (7), applied to some vector $|\psi\rangle \in D_\omega$, is already known to hold for t_α and t'_α , since these are smooth test functions. It follows that Eq. (7) must also hold for our admissible test distributions.

The last statement of the theorem is obvious from the definition of W_n when t and s are smooth functions. By a continuity argument similar to the one above, it also holds for distributional t and s . \square

Since $\mathcal{E}'_n(M, \mathbf{g})$ is a vector space and since Eq. (7) holds, it follows immediately that any $a \in \mathcal{W}(M, \mathbf{g})$ can be written in the form

$$a = t_0 1 + \sum_{n=1}^N W_n(t_n), \quad (13)$$

with $t_0 \in \mathbb{C}$ and $t_n \in \mathcal{E}'_n(M, \mathbf{g})$. Furthermore, the following proposition holds, which will be needed in Sec. 5:

Proposition 2.1. Let $k \geq 0$ and let $a \in \mathcal{W}(M, \mathbf{g})$ be such that

$$[\dots [[a, \varphi(f_1)], \varphi(f_2)], \dots \varphi(f_{k+1})] = 0 \quad \forall f_1, \dots, f_{k+1} \in \mathcal{D}(M). \quad (14)$$

Then a is of the form $a = t_0 1 + \sum_{n=1}^k W_n(t_n)$, where $t_0 \in \mathbb{C}$ and $t_n \in \mathcal{E}'_n(M, \mathbf{g})$.

Proof. a must be of the form (13) where N is some natural number. We must show that $N \leq k$. Let us assume that $N > k$ and that $W_N(t_N) \neq 0$. We show that this leads to a contradiction. By assumption $[\dots [a, \varphi(f_1)], \dots \varphi(f_{N+1})] = 0$ for all test functions. Using Eq. (7) (and recalling that $\varphi(f) = W_1(f)$), this gives us

$$(\Delta_{\mathbf{g}} \otimes \dots \otimes \Delta_{\mathbf{g}}) t_N(x_1, \dots, x_N) \equiv 0. \quad (15)$$

Using the relation $\Delta_{\mathbf{g}} = \Delta_{\mathbf{g}}^{\text{adv}} - \Delta_{\mathbf{g}}^{\text{ret}}$, the support properties of the advanced and retarded fundamental solutions and the fact that t_N is compactly supported, one finds from Eq.

(15) that the distribution $s = (\Delta_{\mathbf{g}}^{\text{ret}} \otimes \cdots \otimes \Delta_{\mathbf{g}}^{\text{ret}})t_N$ must be of compact support. In combination with a microlocal argument similar to the one given in the proof of Thm. 2.1, one finds moreover that $s \in \mathcal{E}'_N(M, \mathbf{g})$. Since $t_N(x_1, \dots, x_N) = \prod_{i=1}^N (\square_{\mathbf{g}} - \xi R_{\mathbf{g}} - m^2)_{x_i} s(x_1, \dots, x_N)$, it follows from Thm. 2.1 that $W_N(t_N) = 0$, which contradicts our hypothesis. \square

That the algebra $\mathcal{W}(M, \mathbf{g})$ contains normal ordered Wick products can be seen as follows. Let

$$t(x_1, \dots, x_k) = f(x_1)\delta(x_1, \dots, x_k), \quad f \in \mathcal{D}(M). \quad (16)$$

The distribution t is in $\mathcal{E}'_k(M, \mathbf{g})$, because

$$\text{WF}(t) = \{(x, k_1, \dots, x, k_k) \in (T^*M)^k \setminus \{0\} \mid \sum_i k_i = 0\} \subset \mathbf{G}_k(M, \mathbf{g}).$$

The algebraic element $W_k(t)$ with t as in Eq. (16) is then just the n -th normal ordered Wick power of a free field operator, as previously defined in [1],

$$:\varphi^k(f):_{\omega} = W_k(t). \quad (17)$$

More generally, we may take t to be

$$t(x_1, \dots, x_r) = \delta(x_{i_1}, \dots) \dots \delta(x_{i_n}, \dots) f_1(x_{i_1}) \dots f_r(x_{i_n}) \quad (18)$$

where $I_1 = \{i_1, \dots\}, \dots, I_n = \{i_n, \dots\}$ is a partition of $\{1, \dots, r\}$ into n pairwise disjoint subsets with $|I_j| = k_j$. This gives us the generalized Wick product

$$:\varphi^{k_1}(f_1) \dots \varphi^{k_n}(f_n):_{\omega} = W_r(t). \quad (19)$$

As was shown in [2], $\mathcal{W}(M, \mathbf{g})$ also contains time ordered products of Wick-powers of free fields.

We next discuss the dependence of the algebra $\mathcal{W}(M, \mathbf{g})$ on our choice of a reference state ω . Let us suppose we had started with another quasi-free Hadamard state ω' . We would then have obtained another algebra $\mathcal{W}'(M, \mathbf{g})$ generated by corresponding operators acting on the GNS Hilbert space constructed from ω' . If the GNS representations of ω and ω' were unitarily equivalent, then the Bogoliubov transformation implementing that unitary equivalence would induce a canonical isomorphism between $\mathcal{W}(M, \mathbf{g})$ and $\mathcal{W}'(M, \mathbf{g})$. However, even if the GNS representations of ω and ω' fail to be unitarily equivalent, at the algebraic level, there is nevertheless a canonical isomorphism:

Lemma 2.1. There is a canonical $*$ -isomorphism $\alpha : \mathcal{W}'(M, \mathbf{g}) \rightarrow \mathcal{W}(M, \mathbf{g})$, which acts on the generators W'_n of $\mathcal{W}'(M, \mathbf{g})$ by

$$\alpha(W'_n(t)) \stackrel{\text{def}}{=} \sum_k W_{n-2k}(\langle d^{\otimes k}, t \rangle), \quad (20)$$

where W_n denote the generators in $\mathcal{W}(M, \mathbf{g})$, and we are using the following notation: $d(x_1, x_2) = \omega(x_1, x_2) - \omega'(x_1, x_2)$ and

$$\langle d^{\otimes k}, t \rangle(x_1, \dots, x_{n-2k}) \stackrel{\text{def}}{=} \frac{n!}{(2k)!(n-2k)!} \int_{M^{2k}} t(y_1, \dots, y_{2k}, x_1, \dots, x_{n-2k}) \times \prod_{i=1}^k d(y_{2i-1}, y_{2i}) \mu_{\mathbf{g}}(y_{2i-1}) \mu_{\mathbf{g}}(y_{2i}) \quad (21)$$

for $2k \leq n$ and $\langle d^{\otimes k}, t \rangle = 0$ for $2k > n$.

Proof. In order to show that the right hand side of Eq. (20) represents an element in $\mathcal{W}(M, \mathbf{g})$, we must show that $\langle d^{\otimes k}, t \rangle \in \mathcal{E}'_{n-2k}(M, \mathbf{g})$. We first note that, since ω and ω' are Hadamard states, d is smooth. By [11, Thm. 8.2.13] we therefore find

$$\begin{aligned} \text{WF}(\langle d^{\otimes k}, t \rangle) &\subset \{(x_1, k_1, \dots, x_{n-2k}, k_{n-2k}) \in (T^*M)^{n-2k} \setminus \{0\} \mid \\ &\quad \exists (x_1, k_1, \dots, x_{n-2k}, k_{n-2k}, y_1, 0, \dots, y_{2k}, 0) \in \mathbf{G}_n(M, \mathbf{g})\} \subset \mathbf{G}_{n-2k}(M, \mathbf{g}). \end{aligned} \quad (22)$$

The distribution $\langle d^{\otimes k}, t \rangle$ is by definition symmetric and of compact support. Therefore $\langle d^{\otimes k}, t \rangle \in \mathcal{E}'_n(M, \mathbf{g})$, which gives us that $\alpha(W_n(t)) \in \mathcal{W}(M, \mathbf{g})$. Since every element in $\mathcal{W}'(M, \mathbf{g})$ can be written as a sum of elements of the form $W'_n(t)$, with $t \in \mathcal{E}'_n(M, \mathbf{g})$, we may therefore take Eq. (20) as the definition of a linear map from $\mathcal{W}'(M, \mathbf{g})$ to $\mathcal{W}(M, \mathbf{g})$. That this map is a homomorphism is demonstrated by the following calculation.

$$\begin{aligned} \alpha(W'_n(t))\alpha(W'_m(t')) &= \sum_{k,l} W_{n-2k}(\langle d^{\otimes k}, t \rangle) W_{n-2l}(\langle d^{\otimes l}, t' \rangle) \\ &= \sum_i \sum_{k,l} W_{n+m-2(k+l+i)}(\langle d^{\otimes k}, t \rangle \otimes_i \langle d^{\otimes l}, t' \rangle) \\ &= \sum_i \sum_r \sum_{k=0}^r W_{n+m-2(r+i)}(\langle d^{\otimes k}, t \rangle \otimes_i \langle d^{\otimes(r-k)}, t' \rangle) \\ &= \sum_i \sum_r W_{n+m-2(r+i)}(\langle d^{\otimes r}, t \otimes_i t' \rangle) \\ &= \alpha(W'_n(t)W'_m(t')), \end{aligned} \quad (23)$$

where we have used the identity

$$\sum_{k=0}^r \langle d^{\otimes k}, t \rangle \otimes_i \langle d^{\otimes(r-k)}, t' \rangle = \langle d^{\otimes r}, t \otimes_i t' \rangle. \quad (24)$$

That α preserves the $*$ -operation follows because d is real, which is in turn a consequence of the fact that $\text{Im } \omega = \text{Im } \omega' = \frac{1}{2}\Delta_{\mathbf{g}}$. That α is one-to-one can be seen from an explicit construction of its inverse, given by the same formula as (20), but with d replaced by $-d$. \square

It should be noted here that the abstract algebra $\mathcal{W}(M, \mathbf{g})$ could be defined more simply and directly as the algebra of expressions of the form Eq. (13), with a product defined by Eq. (7), a $*$ -operation defined by $W_n(t)^* = W_n(\bar{t})$ and which satisfy $W_n(t) = 0$ whenever t is of the form $t(x_1, \dots, x_n) = (\square_{\mathbf{g}} - \xi R_{\mathbf{g}} - m^2)_{x_i} s(x_1, \dots, x_n)$. (Note, however, that the definition of the product (7) requires a choice of Hadamard state ω ; see Eq. (8).) However, our explicit construction of $\mathcal{W}(M, \mathbf{g})$ as an operator algebra on the GNS representation of a quasi-free state, ω , on $\mathcal{A}(M, \mathbf{g})$, is useful for establishing that a suitably wide class of states exists on $\mathcal{W}(M, \mathbf{g})$. In addition, the concrete realization of $\mathcal{W}(M, \mathbf{g})$ will be useful in our explicit construction of local Wick products.

For later purposes, we also need to define a notion of convergence within the algebra $\mathcal{W}(M, \mathbf{g})$. In particular, we would like to have a notion of convergence which is preserved under taking products in our algebra, and which is independent of the quasi-free Hadamard state ω by which this algebra is defined. Such a notion can be defined as follows.

Let $\{t_\alpha\}$ be a sequence of distributions in $\mathcal{E}'_n(M, \mathbf{g})$ with $\text{WF}(t_\alpha) \subset \Gamma_n \forall \alpha$, where Γ_n is some closed conic set contained in $\mathbf{G}_n(M, \mathbf{g})$. Then we say that

$$a_\alpha = W_n(t_\alpha) \rightarrow a = W_n(t) \quad \text{in } \mathcal{W}(M, \mathbf{g})$$

if

$$t_\alpha \rightarrow t \text{ in } \mathcal{D}'_{\Gamma_n}(M^n),$$

i.e., if $t_\alpha \rightarrow t$ in the sense of the Hörmander pseudo topology associated with the cone Γ_n (for the definition of this pseudo topology and the spaces $\mathcal{D}'_{\Gamma_n}(M^n)$ we refer to the Appendix). Convergence in the Hörmander pseudo topology guarantees that $t \in \mathcal{E}'_n(M, \mathbf{g})$. Therefore our algebra is closed with respect to the above notion of convergence. Clearly, that notion is also independent of the particular quasi-free Hadamard state chosen to define $\mathcal{W}(M, \mathbf{g})$. Finally, let $a_\alpha \rightarrow a$ and $b_\alpha \rightarrow b$ be two convergent sequences in $\mathcal{W}(M, \mathbf{g})$ in the above sense. Then, by an argument almost identical to the one given towards the end of the proof of Thm. 2.1, we also have $a_\alpha b_\alpha \rightarrow ab$. Hence, the element-wise product of two convergent sequences of algebraic elements gives again a convergent sequence.

3 Mathematical formulation of the notion of a local, covariant quantum field

The field quantities of interest in quantum field theory in curved spacetime such as the stress energy tensor of free fields or the quantity “ $\lambda\varphi^4$ ” should be local and covariant, i.e., their definition should not depend on structures that are only globally defined (such as a preferred vacuum state) nor should they depend on non-covariant structures (such as a preferred coordinate system). The aim of this section is to explain precisely what we

mean by the statement that an element in $\mathcal{W}(M, \mathbf{g})$ is “locally defined” and “transforms covariantly under diffeomorphisms”. This notion requires the consideration of a given operator on spacetimes (M, \mathbf{g}) and (M', \mathbf{g}') that have isometric regions, but that are not globally isometric. The basic problem is that operators living on (M, \mathbf{g}) and (M', \mathbf{g}') belong to different algebras, and therefore cannot be compared directly. Therefore, we must first provide a natural and consistent identification of the corresponding algebras (see Lem. 3.1). For this purpose, we consider “causality preserving isometric embeddings”, that is, isometric embeddings $\chi : N \rightarrow M$ from a spacetime (N, \mathbf{g}') to another spacetime (M, \mathbf{g}) so that the causal structure on $\chi(N)$ induced from (N, \mathbf{g}') coincides with that induced from (M, \mathbf{g}) . (This is equivalent to the condition that χ preserves the time-orientation and that $J^+(x) \cap J^-(y) \subset \chi(N) \forall x, y \in \chi(N)$.)

Lemma 3.1. Let $\chi : N \rightarrow M$ be an isometric embedding of some globally hyperbolic spacetime (N, \mathbf{g}') into another globally hyperbolic spacetime (M, \mathbf{g}) (so that in fact $\mathbf{g}' = \chi^* \mathbf{g}$) which is causality preserving. Denote by $\mathcal{W}(N, \mathbf{g}')$ and $\mathcal{W}(M, \mathbf{g})$ the corresponding extended Wick-polynomial algebras, viewed as abstract algebras. Then there is a natural injective *-homomorphism $\iota_\chi : \mathcal{W}(N, \mathbf{g}') \rightarrow \mathcal{W}(M, \mathbf{g})$ such that if ω is a quasi-free Hadamard state on (M, \mathbf{g}) and $\omega'(x, y) = \omega(\chi(x), \chi(y))$ we have

$$\iota_\chi(W'_n(t)) = W_n(t \circ \chi^{-1}) \quad \forall t \in \mathcal{E}'_n(N, \mathbf{g}'), \quad (25)$$

where W'_n and W_n are given by Eq. (6) in the GNS representations of ω' and ω respectively and $\chi^{-1} : \chi(N) \rightarrow N$ is the inverse of χ (defined on the image of N under χ).

Proof. Let ω be a quasi-free Hadamard state for the spacetime (M, \mathbf{g}) and let $\omega'(x, y) = \omega(\chi(x), \chi(y))$. Then $\omega'(x, y)$ is the two-point function of a quasi-free Hadamard state ω' on (N, \mathbf{g}') . (Here we are using the assumption that our isometry χ is causality preserving.) By Lem. 2.1, we may assume that the abstract algebras $\mathcal{W}(N, \mathbf{g}')$ and $\mathcal{W}(M, \mathbf{g})$ are concretely realized as linear operators on the GNS constructions of the quasi-free Hadamard states ω' and ω . Since every element in $\mathcal{W}(N, \mathbf{g}')$ can be written as a sum of elements of the form $W'_n(t)$, the above formula gives, by linearity, a map from $\mathcal{W}(N, \mathbf{g}')$ to $\mathcal{W}(M, \mathbf{g})$. That this map is a *-homomorphism can easily be seen from the formulas (7) and (8), together with the relation $\omega'(x, y) = \omega(\chi(x), \chi(y))$. That ι_χ is injective follows from the definition. \square

Remarks: (1) If ω'' is an arbitrary quasi-free Hadamard state on (N, \mathbf{g}') , then, in terms of the generators $W''_n(t)$ of $\mathcal{W}(N, \mathbf{g}')$ in the GNS representation of ω'' , we have

$$\iota_\chi(W''_n(t)) = \sum_k W_{n-2k}(\langle d_\chi^{\otimes k}, t \rangle \circ \chi^{-1}), \quad (26)$$

where $d_\chi(x, y) = \omega(\chi(x), \chi(y)) - \omega''(x, y)$ and where $\langle d_\chi^{\otimes k}, t \rangle$ is given by Eq. (21).

(2) We note that the identifications provided by the maps ι_χ are consistent in the following sense. Let $\chi_{1,2} : M_1 \rightarrow M_2$ and $\chi_{2,3} : M_2 \rightarrow M_3$ be causality preserving

isometric embeddings and $\chi_{1,3} = \chi_{2,3} \circ \chi_{1,2}$. Then the corresponding homomorphisms satisfy (in the obvious notation)

$$\iota_{1,3} = \iota_{2,3} \circ \iota_{1,2}.$$

Definition 3.1. A quantum field Φ (in one variable) is an assignment which associates with every globally hyperbolic spacetime (M, \mathbf{g}) a distribution $\Phi[\mathbf{g}]$ taking values in the algebra $\mathcal{W}(M, \mathbf{g})$, i.e., a continuous linear map $\Phi[\mathbf{g}] : \mathcal{D}(M) \rightarrow \mathcal{W}(M, \mathbf{g})$.

Using the identifications provided by Lem. 3.1, we can now state what we mean by Φ being a “local, covariant quantum field”.

Definition 3.2. A quantum field Φ (in one variable) is said to be *local and covariant*, if it satisfies the following property: Let χ be an isometric embedding map from a spacetime (N, \mathbf{g}') into another spacetime (M, \mathbf{g}) (so that in fact $\mathbf{g}' = \chi^* \mathbf{g}$) which is causality preserving. Let $\iota_\chi : \mathcal{W}(N, \mathbf{g}') \rightarrow \mathcal{W}(M, \mathbf{g})$ be the corresponding homomorphism, defined in Lem. 3.1. Then

$$\iota_\chi(\Phi[\chi^* \mathbf{g}](f)) = \Phi[\mathbf{g}](f \circ \chi^{-1}) \quad \text{for all } f \in \mathcal{D}(N). \quad (27)$$

Local fields in n variables are defined in a similar manner. We will sometimes omit the explicit dependence of the fields on the metric.

Remarks: (1) The above type of algebraic formulation of the locality/covariance property was suggested to us by K. Fredenhagen [9]. It is closely related to a formulation of “locality” previously given in [20, pp. 89–91] for the stress energy operator. Antecedents to this idea can be found in [22] and [15].

(2) It should be noted that the above definition involves actually two logically distinct requirements, namely (a) that the quantum field $\Phi[\mathbf{g}]$ under consideration be given by a diffeomorphism covariant expression, and (b) that it be *locally* constructed from the metric. The second requirement is incorporated in the possibility to consider isometries χ which map a spacetime N into a portion of a “larger” spacetime M . This allows one to contemplate a situation in which “the metric is varied outside some globally hyperbolic subset N of a spacetime M ”. Note that the “covariance” axiom of Dimock [5] effectively corresponds to property (a), but since his axiom applies only to *global* isometries, it does not impose the requirement that the field depends only *locally* on the metric (property (b)).

(3) To illustrate our notion of local, covariant fields and to show that locality is in fact not a trivial requirement, we now display an example of a field which fails to be local. We consider, for every spacetime (M, \mathbf{g}) , the operator-valued distribution $\Phi[\mathbf{g}] = : \varphi^2 :_{\omega_{(M, \mathbf{g})}}$, viewed now as an element of the abstract algebra $\mathcal{W}(M, \mathbf{g})$, where $\omega_{(M, \mathbf{g})}$ is a quasi-free Hadamard state. We claim that the field Φ is not a local, covariant field, no matter how one assigns states $\omega_{(M, \mathbf{g})}$ with globally hyperbolic spacetimes (M, \mathbf{g}) . The crucial

observation needed to prove this is that the locality requirement, Def. 3.2, would imply the following consistency relation between the two-point functions of the given family of quasi-free Hadamard states:

$$\omega_{(M,\mathbf{g})}(\chi(x), \chi(y)) = \omega_{(N,\mathbf{g}')} (x, y) \quad \forall (x, y) \in N \times N, \quad (28)$$

whenever $\chi : N \rightarrow M$ is an isometric embedding map between two spacetimes (N, \mathbf{g}') and (M, \mathbf{g}) (so that in fact $\mathbf{g}' = \chi^* \mathbf{g}$). To see that it is impossible to satisfy this constraint, consider the spacetimes (M, \mathbf{g}) and (M, \mathbf{g}') such that $\mathbf{g} \equiv \mathbf{g}'$ everywhere outside some region O with compact closure. Let $\omega_{(M,\mathbf{g})}$ and $\omega_{(M,\mathbf{g}')}$ be the quasi-free Hadamard states associated with those spacetimes. Let us now choose a Cauchy surface Σ_+ to the future of O and a Cauchy surface Σ_- to the past of O . Furthermore let us choose globally hyperbolic neighborhoods N_\pm of Σ_\pm , which do not intersect O . The consistency requirement, Eq. (28), applied to the embeddings of (N_\pm, \mathbf{g}) into the spacetimes (M, \mathbf{g}) resp. (M, \mathbf{g}') then immediately gives that $\omega_{(M,\mathbf{g})}(x, y) = \omega_{(N_\pm, \mathbf{g})}(x, y)$ for all $(x, y) \in N_\pm \times N_\pm$ and that $\omega_{(M,\mathbf{g}')} (x, y) = \omega_{(N_\pm, \mathbf{g}')} (x, y)$ for all $(x, y) \in N_\pm \times N_\pm$. From this we get

$$\omega_{(M,\mathbf{g})}(x, y) = \omega_{(M,\mathbf{g}')} (x, y) \quad \forall (x, y) \in N_+ \times N_+ \text{ and } \forall (x, y) \in N_- \times N_-. \quad (29)$$

This means that the two-point functions of the states $\omega_{(M,\mathbf{g})}$ and $\omega_{(M,\mathbf{g}')}$ have the same initial data both on Σ_+ and Σ_- . But they do not obey the same field equation (the metrics \mathbf{g} and \mathbf{g}' being different inside O). From this one can easily obtain a contradiction.

The above argument can be applied to any normal ordered operator, in particular to the normal ordered stress energy tensor. Our argument therefore gives a precise meaning to the common statement that normal ordering is not a valid procedure for defining the quantum stress-energy tensor in curved spacetime: The normal ordered stress tensor is not a local, covariant field.

For later purposes, we also find it useful to make the following definition.

Definition 3.3. Let $\Phi(x_1, \dots, x_n)$ be a local, covariant field in n variables. Then, for any globally hyperbolic spacetime, (M, \mathbf{g}) , we define a conic subset $\Gamma^\Phi(M, \mathbf{g}) \subset (T^*M)^n \setminus \{0\}$ associated with Φ by

$$\Gamma^\Phi(M, \mathbf{g}) \stackrel{\text{def}}{=} \overline{\bigcup_{\omega} \text{WF}(\omega(\Phi[\mathbf{g}](\cdot)))}, \quad (30)$$

where the closure is taken in $(T^*M)^n \setminus \{0\}$, and where the union runs over all quasi-free Hadamard states.

Remark: If χ is a global diffeomorphism of M , then we have $\Gamma^\Phi(M, \chi^* \mathbf{g}) = \chi^* \Gamma^\Phi(M, \mathbf{g})$. This is a straightforward consequence of our notion of local, covariant fields.

4 Additional properties of local Wick polynomials and their time ordered products

As we have seen, although normal ordering is mathematically a well defined prescription for defining powers of field operators, it does not define a local, covariant field, and is therefore not of any particular physical interest. Consequently, the same also applies to time ordered products of normal ordered Wick powers. In particular, the latter should not be used for the perturbative definition of an interacting field theory, since this field theory would then depend on nonlocal information, namely the global properties of the state chosen for the normal ordering prescription. We therefore seek to define a notion of *local* Wick polynomials and *local* time ordered products in the algebras $\mathcal{W}(M, \mathbf{g})$. In the present section, we shall specify these fields axiomatically (but not uniquely, as we shall see) by certain properties, which can heuristically be stated as follows:

- (i) **Locality:** The sought-for Wick products and time ordered products are local, covariant fields in the sense of Def. 3.2.
- (ii) **Specific properties:** They have properties analogous to certain properties known to hold for the normal ordered Wick products and the time ordered products of these, such as for example a specific expression for their commutator with a free field.
- (iii) **Continuity and Analyticity:** The fields vary analytically (continuously) under analytic (smooth) variations of the metric and the coupling parameters.
- (iv) **Scaling:** The fields scale homogeneously “up to logarithmic terms” under a rescaling of the metric and the coupling parameters.

We have given a precise definition of requirement (i) in the previous section. A mathematically precise formulation of conditions (ii)—(iv) will now be given in the following three subsections.

4.1 Specific properties

We first consider local Wick powers of the free field without derivatives. These are denoted by φ^k , where $k \in \mathbb{N}$. We make the obvious requirement that φ^1 be identical with the free field φ (which is easily checked to be a local, covariant field), and for later convenience we also set $\varphi^0 = 1$. We impose the following conditions on φ^k :

Expansion: $[\varphi^k(x), \varphi(y)] = ik\Delta_{\mathbf{g}}(x, y)\varphi^{k-1}(x)$.

Hermiticity: $\varphi^k(f)^* = \varphi^k(\bar{f})$ for all $f \in \mathcal{D}(M)$.

Microlocal spectrum condition: Let ω be a quasi-free Hadamard state. Then $\omega(\varphi^k(x))$ is a smooth function in x .

Local Wick powers of differentiated fields are required to satisfy suitably generalized versions of the above requirements. The modifications are straightforward and therefore left to the reader. For notational simplicity we will explicitly consider only the undifferentiated Wick powers in the following, but our existence and uniqueness arguments and results apply to the differentiated Wick powers as well as to the undifferentiated Wick powers.

Remark: For the local Wick products of differentiated fields it also would be reasonable to impose the following additional requirement: Any local Wick product containing $(\square - \xi R - m^2)\varphi$ as a factor should vanish. We note that the explicit construction of local Wick products that will be given in Sec. 5.2 does not satisfy that requirement. (A related difficulty with our prescription given in Sec. 5.2 is that it gives a stress energy operator which is not conserved.) We believe that a construction of local Wick products of differentiated fields satisfying this additional condition can be given via the use of the local vacuum-concept introduced by Kay [16] (see also [12, Ch. 6]), but we will defer the consideration of this issue to a future investigation.

We next consider local time ordered products of undifferentiated local Wick powers. These are denoted by $T(\varphi^{k_1} \dots \varphi^{k_n})$. We make the obvious requirement that $T(\varphi^k)$ be equal to the local Wick power φ^k considered above. Our further requirements are the following

Symmetry: Any time ordered product is symmetric under a permutation of the operators under the time-ordering symbol.

Causal factorization: Consider any set of points $(x_1, \dots, x_n) \in M^n$ and a partition of $\{1, \dots, n\}$ into two non-empty subsets I and I^c , with the property that no point x_i with $i \in I$ is in the past of any of the points x_j with $j \in I^c$, i.e., $x_i \notin J^-(x_j)$ for all $i \in I$ and $j \in I^c$. Then the time ordered products factorize in the following sense:

$$T(\varphi^{k_1}(x_1) \dots \varphi^{k_n}(x_n)) = T\left(\prod_{i \in I} \varphi^{k_i}(x_i)\right) T\left(\prod_{j \in I^c} \varphi^{k_j}(x_j)\right).$$

Expansion: $[T(\varphi^{k_1}(x_1) \dots \varphi^{k_n}(x_n)), \varphi(y)] =$

$$i \sum_{i=1}^n k_i \Delta_{\mathbf{g}}(x_i, y) T(\varphi^{k_1}(x_1) \dots \varphi^{k_{i-1}}(x_i) \dots \varphi^{k_n}(x_n)).$$

Unitarity: $T(\varphi^{k_1}(x_1) \dots \varphi^{k_n}(x_n))^* = \sum_{\mathcal{P}=I_1 \uplus \dots \uplus I_j} (-1)^{n+j} \prod_{I \in \mathcal{P}} T\left(\prod_{i \in I} \varphi^{k_i}(x_i)\right).$

Here we have used the following notation: $\mathcal{P} = I_1 \uplus \dots \uplus I_j$ denotes a partition of the set $\{1, \dots, n\}$ into j pairwise disjoint, nonempty subsets I_i . The unitarity condition is

equivalent to requiring that the S -matrix is unitary in the sense of formal power series of operators.

Microlocal spectrum condition: Let $\Gamma^T(M, \mathbf{g}) \subset (T^*M)^n \setminus \{0\}$ be the conic set associated with the time ordered product $T(\varphi^{k_1}(x_1) \dots \varphi^{k_n}(x_n))$ as in Def. 3.3. Then, any point $(x_1, k_1, \dots, x_n, k_n)$ in $\Gamma^T(M, \mathbf{g})$ satisfies the following: (a) there exist null-geodesics $\gamma_1, \dots, \gamma_m$ which connect any point x_j in the set $\{x_1, \dots, x_n\}$ to some other point in that set, (b) there exists coparallel, cotangent covectorfields p_1, \dots, p_m along these geodesics such that $p_i \in V^+$ if the starting point of γ_i is not in the causal past of the end point of γ_i , (c) for the covector k_j over the point x_j it holds that $k_j = \sum_e p_e(x_j) - \sum_s p_s(x_j)$, where the index e runs through all null-geodesics ending at x_j and s runs through all null-geodesics starting at x_j .

The microlocal spectrum condition may be viewed as a microlocal analogue of translation invariance in Minkowski space. It was shown to hold for time ordered products of normal ordered Wick powers in [2]. We also note that it reduces to the requirement that $\omega(\varphi^k(x))$ be smooth in the case $n = 1$.

Again, time ordered products of differentiated Wick powers would satisfy suitable generalizations of the above requirements. Our uniqueness arguments of Sec. 5.3 would also apply to such time ordered products, but for notational simplicity we shall explicitly only consider the undifferentiated products below.

For later purposes, we also wish to impose a sharpened version of the microlocal spectrum condition for the local Wick polynomials and their time ordered products for the case that the metric \mathbf{g} is not only smooth, but in addition real analytic in some convex normal neighborhood $O \subset M$. For this purpose, we consider “analytic” quasi-free Hadamard states, i.e., quasi-free states ω with the property that $\omega(x, y) - H(x, y)$ is not only a smooth, but in addition an analytic function in $O \times O$, where H is the Hadamard fundamental solution defined by Eq. (4). We then impose a sharpened constraint on the singular behavior of the expectation values of a local time ordered product in such a state by considering the so-called “analytic wave front set” [11] instead of the ordinary, “smooth wave front set”, which is used in the above microlocal spectrum condition (compare Def. 3.3). The concept of the analytic wave front set, $\text{WF}_A(u)$, of a distribution u characterizes the points and directions for which u fails to be analytic, in much the same way as the ordinary wave front set, $\text{WF}(u)$, characterizes the points and directions for which u is not smooth.⁴

In order to give a formulation of the microlocal spectrum condition in the analytic case that is parallel to the one given above in the smooth case, we first introduce, for every local, covariant field $\Phi(x_1, \dots, x_n)$, a conic set $\Gamma_A^\Phi(O, \mathbf{g}) \subset (T^*O)^n \setminus \{0\}$, which is defined as in Def. 3.3, but with the difference that the union in Eq. (30) now runs over all analytic Hadamard states in $O \times O$, and that WF is replaced by WF_A . In the case

⁴We note that for any distribution u it holds that $\text{WF}(u) \subset \text{WF}_A(u)$.

when $\Phi(x_1, \dots, x_n)$ is a local time ordered product, we denote this conic set by $\Gamma_A^T(M, \mathbf{g})$. Our analytic microlocal spectrum condition is then the following:

Analytic microlocal spectrum condition: Let O be a convex normal neighborhood of M . Then any point $(x_1, k_1, \dots, x_n, k_n) \in \Gamma_A^T(O, \mathbf{g})$ has the properties stated in the microlocal spectrum condition for the smooth case.

Remark: For a local Wick product (the case $n = 1$), this condition implies that $\omega(\varphi^k(x))$ is analytic in O for any analytic Hadamard state.

4.2 Continuity and analyticity

The basic difficulty in defining notions of continuous and analytic dependence of a local, covariant field under a corresponding variation of the metric and the parameters is that the fields corresponding to different metrics and parameters are elements of different algebras and hence cannot be compared directly. It is therefore necessary to provide a suitable identification of these elements first. In order to simplify the discussion, we will first consider only variations of the spacetime metric, and keep the coupling constants fixed. We will comment on how to generalize the present discussion to include also variations of the parameters at the end of this subsection.

We first give a notion of the continuous dependence of a local, covariant quantum field on the metric. Here, we consider a situation wherein one is given a family of metrics, $\mathbf{g}^{(s)}$, depending smoothly on some real parameter, s , and differing from each other only within some compact region, O , in the spacetime M . Under these circumstances, we will show in Lem. 4.1 that it is possible to construct isomorphisms between the algebras corresponding to different values of s by identifying the observables in the past (or future) of O . A local, covariant field Φ with a continuous dependence under smooth variations of the metric will then be defined as one for which the family $\Phi[\mathbf{g}^{(s)}]$ depends continuously on s under this identification of the corresponding algebras for all smooth families of metrics $\mathbf{g}^{(s)}$.

A notion of the analytic dependence of a local, covariant field under corresponding variations of the metric is given next. Here, we consider an analytic family, $\mathbf{g}^{(s)}$, of real analytic metrics in some open neighborhood O of M . However, unlike in the case of a *smooth* family of metrics considered above, we now cannot demand that our metrics coincide outside some compact region, because there are no analytic functions with compact support. Consequently, we cannot identify the algebras for different values of s in the same manner as in the smooth case, and we therefore have no obvious means to compare directly a given field for the different metrics $\mathbf{g}^{(s)}$, since these fields belong to different algebras. We will avoid this problem by considering instead a notion of analytic dependence of a field on the metric via its expectation values in an analytic family of quasi-free Hadamard states, $\omega^{(s)}$, corresponding to the metrics $\mathbf{g}^{(s)}$: We shall say that a local, covariant field Φ depends analytically on the metric if the family of expectation values $\omega^{(s)}(\Phi[\mathbf{g}^{(s)}](x_1, \dots, x_n))$ depends, in a suitable sense, analytically on s , for all possible

choices of analytic families of metrics $\mathbf{g}^{(s)}$ and states $\omega^{(s)}$.

Lemma 4.1. Consider two globally hyperbolic spacetimes (M, \mathbf{g}) and (M, \mathbf{g}') , such that $\mathbf{g} \equiv \mathbf{g}'$ everywhere outside some region O with compact closure. Then there exists a *-isomorphism $\tau_{\text{ret}} : \mathcal{W}(M, \mathbf{g}') \rightarrow \mathcal{W}(M, \mathbf{g})$, such that the restriction of τ_{ret} to the subalgebra $\mathcal{W}(M_-, \mathbf{g}')$ with $M_- = M \setminus J^+(O)$ is the identity. Similarly there exists a *-isomorphism $\tau_{\text{adv}} : \mathcal{W}(M, \mathbf{g}') \rightarrow \mathcal{W}(M, \mathbf{g})$, such that the restriction of τ_{adv} to the subalgebra $\mathcal{W}(M_+, \mathbf{g}')$ with $M_+ = M \setminus J^-(O)$ is the identity.

Remark: The isomorphisms τ_{ret} and τ_{adv} are constructed by a suitable identification of the fields in both algebras on a Cauchy surface Σ_- not intersecting the future of O or, respectively, on a Cauchy surface Σ_+ not intersecting the past of O . The particular choice of those Cauchy surfaces is irrelevant for the constructions, so in that sense, τ_{ret} and τ_{adv} are canonical. In the following proof, we will only construct τ_{ret} , the construction of τ_{adv} is completely analogous.

Proof. Let Σ_- be a Cauchy surface not intersecting the future of O and let Σ_+ be a Cauchy surface not intersecting the past of O . Define a bidistribution S on M by

$$S(f_1 \otimes f_2) = \int_{\Sigma_-} (F_1 \nabla_a F_2 - F_2 \nabla_a F_1) n^a d\sigma, \quad (31)$$

where

$$F_1(x) = \int_M \Delta_{\mathbf{g}}(x, y) f_1(y) \mu_{\mathbf{g}}(y), \quad F_2(x) = \int_M \Delta_{\mathbf{g}'}(x, y) f_2(y) \mu_{\mathbf{g}'}(y). \quad (32)$$

By a standard argument based on Gauss' law (see e.g. [20]), one can see that S does not depend on the particular choice for Σ_- . Let χ be an arbitrary smooth function on M satisfying $\chi(x) = 0$ for all $x \in J^+(\Sigma_+)$ and $\chi(x) = 1$ for all $x \in J^-(\Sigma_-)$. We then define a linear map $A_{\text{ret}} : \mathcal{D}(M) \rightarrow \mathcal{D}'(M)$ by

$$A_{\text{ret}} f \stackrel{\text{def}}{=} -(\square_{\mathbf{g}} - \xi R_{\mathbf{g}} - m^2)(\chi S f).$$

The distribution $A_{\text{ret}} f$ satisfies the following properties:

- (a) $A_{\text{ret}} f$ is of compact support with $\text{supp}(A_{\text{ret}} f) \subset J^+(\Sigma_-) \cap J^-(\Sigma_+)$,
- (b) $\Delta_{\mathbf{g}} A_{\text{ret}} f(x) = \Delta_{\mathbf{g}'} f(x)$ for all $x \in J^-(\Sigma_-)$ and $f \in \mathcal{D}(M)$.

Item (a) immediately follows from the fact that $(\square_{\mathbf{g}} - \xi R_{\mathbf{g}} - m^2)S f(x) = 0$ for all $x \in J^-(\Sigma_-)$ and the fact that $\chi(x) = 0$ for all $x \in J^+(\Sigma_+)$. Item (b) holds since

$$\Delta_{\mathbf{g}} A_{\text{ret}} f(x) = \Delta_{\mathbf{g}}^{\text{ret}}(\square_{\mathbf{g}} - \xi R_{\mathbf{g}} - m^2)(\chi S f)(x) = S f(x) = \Delta_{\mathbf{g}'} f(x) \quad \forall x \in J^-(\Sigma_-). \quad (33)$$

We wish to show that the n -th tensor power of A_{ret} gives a map

$$A_{\text{ret}}^{\otimes n} : \mathcal{E}'_n(M, \mathbf{g}') \rightarrow \mathcal{E}'_n(M, \mathbf{g}).$$

We begin by showing that S has the following wave front set:

$$\begin{aligned} \text{WF}(S) \subset & \{(x_1, k_1, x_2, -k_2) \in (T^*M)^2 \setminus \{0\} \mid \exists y \in M \setminus J^+(O) \text{ and } (y, p) \in T_y^*M \\ & \text{such that } (x_1, k_1) \sim (y, p) \text{ with respect to } \mathbf{g} \text{ and such that} \\ & (x_2, k_2) \sim (y, p) \text{ with respect to } \mathbf{g}'\}. \end{aligned} \quad (34)$$

In order to see this, we note that by definition,

$$(\square_{\mathbf{g}} - \xi R_{\mathbf{g}} - m^2)_x S(x, y) = (\square_{\mathbf{g}'} - \xi R_{\mathbf{g}'} - m^2)_y S(x, y) = 0. \quad (35)$$

We are thus in a position to apply the “propagation of singularities theorem” [7, Thm. 6.1.1] to S . This theorem tells us that an element (x_1, k_1, x_2, k_2) is in $\text{WF}(S)$ if and only if every element of the form (y_1, p_1, y_2, p_2) is in $\text{WF}(S)$, where $(y_1, p_1) \sim (x_1, k_1)$ with respect to \mathbf{g} and where $(y_2, p_2) \sim (x_2, k_2)$ with respect to \mathbf{g}' . Moreover, by definition of S , we have that $S(x, y) = \Delta_{\mathbf{g}}(x, y) = \Delta_{\mathbf{g}'}(x, y)$ for all $x, y \in M \setminus J^+(O)$. The wave front set of $\Delta_{\mathbf{g}}$ is known to be

$$\text{WF}(\Delta_{\mathbf{g}}) = \{(x_1, k_1, x_2, -k_2) \in (T^*M)^2 \setminus \{0\} \mid (x_1, k_1) \sim (x_2, k_2) \text{ with respect to } \mathbf{g}\}. \quad (36)$$

Combining these two pieces of information then gives us the above wave front set for S .

Since differentiating and multiplying a distribution by a smooth function does not enlarge its wave front set, it holds that $\text{WF}(A_{\text{ret}}) \subset \text{WF}(S)$. By the rules [11] for calculating the wave front set of a tensor product of distributions, we get from this that

$$\begin{aligned} \text{WF}(A_{\text{ret}}^{\otimes n}) \subset & \{(x_1, k_1, \dots, x_n, k_n, y_1, p_1, \dots, y_n, p_n) \in (T^*M)^{2n} \setminus \{0\} \mid \\ & (x_i, k_i, y_i, p_i) \in \text{WF}(S) \cup \{0\} \text{ for all } i = 1, \dots, n\}. \end{aligned} \quad (37)$$

Let $t \in \mathcal{E}'_n(M, \mathbf{g}')$, that is, t is a symmetric, compactly supported n -point distribution with $\text{WF}(t) \subset \mathbf{G}_n(M, \mathbf{g}')$. Then it follows from the above form of $\text{WF}(A_{\text{ret}}^{\otimes n})$ that

$$\begin{aligned} & \{(y_1, p_1, \dots, y_n, p_n) \in (T^*M)^n \setminus \{0\} \mid \\ & \exists (x_1, 0, \dots, x_n, 0, y_1, -p_1, \dots, y_n, -p_n) \in \text{WF}(A_{\text{ret}}^{\otimes n})\} \cap \text{WF}(t) = \emptyset. \end{aligned} \quad (38)$$

Therefore [11, Thm. 8.2.13] applies and we conclude from that theorem that the linear operator $A_{\text{ret}}^{\otimes n}$ has a well-defined action on distributions $t \in \mathcal{E}'_n(M, \mathbf{g}')$. The wave front set of the distribution $A_{\text{ret}}^{\otimes n} t$ can be calculated from [11, Thm. 8.2.13] using our knowledge about $\text{WF}(A_{\text{ret}}^{\otimes n})$ and $\text{WF}(t)$:

$$\begin{aligned} \text{WF}(A_{\text{ret}}^{\otimes n} t) \subset & \{(x_1, k_1, \dots, x_n, k_n) \in (T^*M)^n \setminus \{0\} \mid \exists (x_i, k_i, y_i, -p_i) \in \text{WF}(S) \cup \{0\}, \\ & i = 1, \dots, n, \text{ such that } (y_1, p_1, \dots, y_n, p_n) \in \mathbf{G}_n(M, \mathbf{g}')\} \\ \cup & \{(x_1, k_1, \dots, x_n, k_n) \in (T^*M)^n \setminus \{0\} \mid \exists (x_i, k_i, y_i, 0) \in \text{WF}(S) \cup \{0\} \\ & \text{for all } i = 1, \dots, n\} \\ \subset & \mathbf{G}_n(M, \mathbf{g}). \end{aligned} \quad (39)$$

Since the distribution $A_{\text{ret}}^{\otimes n} t$ is of compact support by (a), we have thus demonstrated that the n -th tensor power of A_{ret} gives a map from $\mathcal{E}'_n(M, \mathbf{g}')$ to $\mathcal{E}'_n(M, \mathbf{g})$, as we had claimed.

The algebras $\mathcal{W}(M, \mathbf{g})$ and $\mathcal{W}(M, \mathbf{g}')$ are faithfully represented on the GNS Hilbert spaces of any quasi-free Hadamard states ω respectively ω' on the corresponding Weyl subalgebras. We may choose these quasi-free states (or rather their two-point functions) to have identical initial data on Σ_- . In view of item (b), this amounts to saying that

$$\omega(A_{\text{ret}} f_1 \otimes A_{\text{ret}} f_2) = \omega'(f_1 \otimes f_2) \quad (40)$$

for all compactly supported test functions f_1, f_2 . We now define $\tau_{\text{ret}} : \mathcal{W}(M, \mathbf{g}') \rightarrow \mathcal{W}(M, \mathbf{g})$ by

$$\tau_{\text{ret}}(W'_n(t)) \stackrel{\text{def}}{=} W_n(A_{\text{ret}}^{\otimes n} t), \quad (41)$$

where the W'_n are the generators of $\mathcal{W}(M, \mathbf{g}')$ and where the W_n are the generators of $\mathcal{W}(M, \mathbf{g})$. We must show that this is indeed a *-isomorphism. That A_{ret} respects the product in both algebras, Eq. (7), follows from

$$A_{\text{ret}}^{\otimes(n+m-2k)}(t \otimes'_k t') = (A_{\text{ret}}^{\otimes m} t) \otimes_k (A_{\text{ret}}^{\otimes n} t'), \quad (42)$$

where ω' is used for the contractions in \otimes'_k on the left side, and ω is used for the contractions in \otimes_k on the right side, as one can easily verify using relation Eq. (40) and the definition of the contracted tensor product. That τ_{ret} respects the *-operation follows because A_{ret} is real. That τ_{ret} is invertible can be seen by an explicit construction of its inverse, given by the same construction as above, but with the spacetimes (M, \mathbf{g}) and (M, \mathbf{g}') interchanged. The definition of A_{ret} does not depend on the specific choice for Σ_- , but it depends on a choice for χ . It is however not difficult to see that isomorphism τ_{ret} itself is independent of that choice. We finally prove that the restriction of τ_{ret} to $\mathcal{W}(M_-, \mathbf{g}')$ is the identity. By item (b) above we have

$$\Delta_{\mathbf{g}}(t - A_{\text{ret}} t) = \Delta_{\mathbf{g}} t - \Delta_{\mathbf{g}'} t \quad \text{in } J^-(\Sigma_-) \quad (43)$$

for any $t \in \mathcal{E}'_1(M, \mathbf{g}')$. Now if the support of t is in M_- (so that $\text{supp}(t) \cap J^+(O) = \emptyset$) then the above expression vanishes on $J^-(\Sigma_-)$. Since this expression is moreover a solution to the Klein-Gordon equation, it must in fact vanish everywhere. Therefore, by the same argument as in the proof of Prop. 2.1, there is an $s \in \mathcal{E}'_1(M, \mathbf{g})$ such that $t - A_{\text{ret}} t = (\square_{\mathbf{g}} - \xi R_{\mathbf{g}} - m^2)s$. Since $W_1((\square_{\mathbf{g}} - \xi R_{\mathbf{g}} - m^2)s) = 0$, this implies that $\tau_{\text{ret}}(W'_1(t)) = W_1(A_{\text{ret}} t) = W_1(t)$ for all $t \in \mathcal{E}'_1(M_-, \mathbf{g}')$. This argument can be generalized to show that $\tau_{\text{ret}}(W'_n(t)) = W_n(t)$ for all $t \in \mathcal{E}'_n(M_-, \mathbf{g}')$ and arbitrary n , thus proving our claim. \square

Using the above lemma, we are now able to say what precisely we mean by the statement that a “local field varies continuously under a smooth variation of the metric”. Let

$\mathbf{g}^{(s)}$ be a family of metrics on M such that $\mathbf{g}^{(s)} \equiv \mathbf{g}$ outside a compact region O and which depends smoothly on s in the sense that the five-dimensional metric $g_{ab}^{(s)} + (ds)_a(ds)_b$ is smooth on $M \times \mathbb{R}$. From the above lemma, we then get, for each value of s , an isomorphism $\tau_{\text{ret}} : \mathcal{W}(M, \mathbf{g}^{(s)}) \rightarrow \mathcal{W}(M, \mathbf{g})$.

Continuity: A local, covariant quantum field Φ is said to depend continuously on the metric if the algebra-valued function

$$\mathbb{R} \ni s \rightarrow \tau_{\text{ret}}\left(\Phi[\mathbf{g}^{(s)}](f)\right) \in \mathcal{W}(M, \mathbf{g})$$

is continuous for all families of metrics as described above and all test functions f .

Remarks: (1) A notion of continuous dependence of the fields on the metric could also be given based on the isomorphisms τ_{adv} . It can be seen (although we do not demonstrate this here) that both notions coincide.

(2) We also note that the isomorphisms τ_{adv} and τ_{ret} can be used in certain cases to describe in a meaningful way the advanced and retarded response of local, covariant quantum field to an infinitesimal perturbation of the metric in the past and future. Namely, for a local, covariant field Φ which has not only a continuous but in addition a once differentiable dependence on the metric, one can define its advanced response, $(\delta\Phi/\delta g_{ab})_{\text{adv}}$, to a metric perturbation by

$$\int_{M^{n+1}} \left(\frac{\delta\Phi(x_1, \dots, x_n)}{\delta g_{ab}(y)} \right)_{\text{adv}} h_{ab}(y) f(x_1, \dots, x_n) \mu_{\mathbf{g}}(y) \mu_{\mathbf{g}}(x_1) \dots \mu_{\mathbf{g}}(x_n) \\ \stackrel{\text{def}}{=} \frac{d}{ds} \tau_{\text{adv}}(\Phi[\mathbf{g} + s\mathbf{h}](f)) \Big|_{s=0}, \quad (44)$$

where $\mathbf{h} \equiv h_{ab}$ is of compact support. In the same way one can define the retarded response, $(\delta\Phi/\delta g_{ab})_{\text{ret}}$, of a local, covariant field Φ to a metric perturbation.

We next explain what we mean by the statement that a “local field varies analytically under an analytic variation of the metric”. Let $\mathbf{g}^{(s)}$ be a family of metrics on M which is analytic in some convex normal neighborhood $O \subset M$ in the sense that the five-dimensional metric $g_{ab}^{(s)} + (ds)_a(ds)_b$ is analytic on $O \times (-\epsilon, \epsilon)$. We consider a family of quasi-free Hadamard states, $\omega^{(s)}$, on the algebras $\mathcal{W}(M, \mathbf{g}^{(s)})$ that is analytic in s in the following sense: Let $H^{(s)}$ be the Hadamard parametrices, given by Eq. (4), constructed from the metrics $\mathbf{g}^{(s)}$, and let us assume that O is small enough such that $H^{(s)}$ is well-defined on $O \times O$ for all s . We say that $\omega^{(s)}$ is an analytic one-parameter family of states if the difference $\omega^{(s)}(x, y) - H^{(s)}(x, y)$ is jointly analytic in (x, y, s) on $O \times O \times (-\epsilon, \epsilon)$. We would like to define a notion of the analytic dependence of a local field on the metric by demanding that the expectation values $\omega^{(s)}(\Phi[\mathbf{g}^{(s)}](x_1, \dots, x_n))$ depend analytically on s for any analytic family of metrics and any corresponding analytic family of quasi-free Hadamard states. However, since these expectation values are in fact distributions in

x_1, \dots, x_n , it is not clear a priori what is actually meant by “analytic dependence on s ”. To give precise meaning to this statement we must characterize the extent to which the above expectation values, viewed as distributions jointly in (x_1, \dots, x_n, s) , “fail to be analytic”. We do so by means of the analytic wave front set of the above expectation values of a local, covariant field, viewed as a distribution jointly in (x_1, \dots, x_n, s) .

Analytic dependence: Let $\mathbf{g}^{(s)}$ be an analytic family of metrics in $O \subset M$ and let $\omega^{(s)}$ be a corresponding analytic family of quasi-free Hadamard states. Let Φ be a local, covariant field in n variables, and let $\Gamma_A^\Phi(O, \mathbf{g}) \subset (T^*O)^n \setminus \{0\}$ be the associated conic set as introduced in Subsection 4.1. Consider the family of expectation values,

$$E_\omega^\Phi(x_1, \dots, x_n, s) \stackrel{\text{def}}{=} \omega^{(s)}\left(\Phi[\mathbf{g}^{(s)}](x_1, \dots, x_n)\right), \quad (45)$$

viewed as a distribution on $O^n \times (-\epsilon, \epsilon)$. Then we demand that

$$\text{WF}_A(E_\omega^\Phi) \subset \{(x_1, k_1, \dots, x_n, k_n, s, \rho) \in T^*(O^n \times (-\epsilon, \epsilon)) \mid (x_1, k_1, \dots, x_n, k_n) \in \Gamma_A^\Phi(O, \mathbf{g}^{(s)})\} \quad (46)$$

for all analytic families of metrics and all corresponding analytic families of states.

Remarks: (1) The above condition on the analytic wave front set can be understood as follows. Consider first an open neighborhood $U \subset O^n$ such that E_ω^Φ is non-singular for all $(x_1, \dots, x_n) \in U$ for a given value of $s = s_0$. Then the condition on $\text{WF}_A(E_\omega^\Phi)$ implies that E_ω^Φ varies analytically in (x_1, \dots, x_n) and s in neighborhood of the form $U \times (s_0 - \delta, s_0 + \delta)$ for some $\delta > 0$. On the other hand, if (x_1, \dots, x_n) is a singular point for the local, covariant field Φ at a given s , then the condition on $\text{WF}_A(E_\omega^\Phi)$ demands that the singular “ x -directions” of E_ω^Φ in momentum space are the same ones as for the field $\Phi[\mathbf{g}^{(s)}](x_1, \dots, x_n)$, considered as a distribution in the x -variables at fixed s .

(2) The above definition assumes the existence of an analytic family of states for any given analytic family of metrics. While we do not have any argument proving the existence of such a family, we remark that, for the sake of our definition of analytic dependence, it would be entirely sufficient to have a suitable family, $\psi^{(s)}$, of normalized, linear (but not necessarily positive) functionals on the algebras $\mathcal{W}(M, \mathbf{g}^{(s)})$. We now briefly indicate how such a family can be constructed. Firstly, using the results of [12, Ch. 6] one can obtain families of bidistributions $\psi^{(s)}(x, y)$ which have the same properties as $\omega^{(s)}(x, y)$, except possibly for positivity. These bidistributions can then be promoted, by the same formula as Eq. (3), to normalized linear functionals on the algebras $\mathcal{A}(M, \mathbf{g}^{(s)})$ of free fields. It is then not difficult to see that these can then be extended (via normal ordering elements of $\mathcal{W}(M, \mathbf{g}^{(s)})$ with respect to $\psi^{(s)}$) to functionals on the algebras $\mathcal{W}(M, \mathbf{g}^{(s)})$.

The analyticity of local, covariant fields under corresponding variations of the coupling parameters can be formulated in a very similar way as above. To obtain a corresponding notion of continuous dependence, it is however necessary to allow the coupling parameters

$p \equiv (\xi, m^2)$ in the case of a real scalar field, Eq. (1)) to be arbitrary smooth functions on spacetime, rather than constants. One can then consider two coupling functions p_1 and p_2 which differ only within some compact region. In such a situation, it is possible to find an identification of the algebras corresponding to p_1 and p_2 , which is analogous to the one established in Lem. 4.1. Based on such an identification, one can give a notion of continuity of local, covariant fields under smooth variations of the coupling parameters, which is completely analogous to the above notion of continuity under smooth variations of the metric. It should also be noted that the consideration of different coupling parameters involves a slight generalization of our notion of local, covariant fields (Def. 3.2). This generalization is however rather obvious and therefore left to the reader.

4.3 Scaling

The scaling requirement involves the comparison of a given local, covariant field at different scales, i.e., its behavior under a rescaling $\mathbf{g} \rightarrow \lambda^{-2}\mathbf{g}$ and under corresponding rescalings of the coupling parameters m^2, ξ and φ , chosen in such a way as to leave the action \mathcal{S} invariant. For the action (1), the unique corresponding scalings of m^2, ξ and φ leaving \mathcal{S} invariant are $m^2 \rightarrow \lambda^2 m^2, \xi \rightarrow \xi$ and $\varphi \rightarrow \lambda\varphi$. We will refer to the various exponents of λ as the “engineering dimension” of the corresponding quantities (and similarly for other quantities derived from those). In order to compare an arbitrary local, covariant field Φ in the algebras $\mathcal{W}(M, \mathbf{g})$ at different scales, we first show that the algebras constructed from the rescaled quantities are naturally isomorphic for all values of $\lambda > 0$.

Lemma 4.2. There are natural $*$ -isomorphisms $\sigma_\lambda : \mathcal{W}_{p(\lambda)}(M, \lambda^{-2}\mathbf{g}) \rightarrow \mathcal{W}_p(M, \mathbf{g})$ for all $\lambda > 0$, where the subscripts on the algebras indicate the dependence on the parameters, $p = (\xi, m^2)$ and $p(\lambda) = (\xi, \lambda^2 m^2)$.

Proof. Let ω be a quasi-free Hadamard state for the theory at $\lambda = 1$. For all $\lambda > 0$, let

$$\omega^{(\lambda)}(x, y) = \lambda^2 \omega(x, y). \quad (47)$$

Then $\omega^{(\lambda)}$ is the two-point function of a quasi-free Hadamard state of the theory scaled by λ . (Note that Eq. (47) is equivalent to the relation $\omega^{(\lambda)}(f_1 \otimes f_2) = \lambda^{-6} \omega(f_1 \otimes f_2)$ between the smeared two-point functions, because the metric volume element transforms as $\mu_{\lambda^{-2}\mathbf{g}} = \lambda^{-4} \mu_{\mathbf{g}}$.) We use $\omega^{(\lambda)}$ to give a concrete realization of the algebra $\mathcal{W}_{p(\lambda)}(M, \lambda^{-2}\mathbf{g})$. We then define (using the same symbol for the generators W_n in both algebras)

$$\sigma_\lambda : \mathcal{W}_{p(\lambda)}(M, \lambda^{-2}\mathbf{g}) \ni W_n(t) \rightarrow \lambda^{-3n} W_n(t) \in \mathcal{W}_p(M, \mathbf{g}).$$

σ_λ is a well defined map for all $\lambda > 0$, because $\mathcal{E}'_n(M, \mathbf{g}) = \mathcal{E}'_n(M, \lambda^{-2}\mathbf{g})$. Using Eq. (47), it is also easily checked to be a $*$ -homomorphism. \square

Using the above lemma, we are now in a position to consider a given local, covariant field at different scales: Let Φ be a local, covariant field in n variables. We then define a rescaled field, $S_\lambda\Phi$, by

$$S_\lambda\Phi[\mathbf{g}, p](f) \stackrel{\text{def}}{=} \lambda^{4n} \sigma_\lambda \left(\Phi[\lambda^{-2}\mathbf{g}, p(\lambda)](f) \right), \quad (48)$$

where $p(\lambda) = (\xi, \lambda^2 m^2)$ and $\lambda > 0$. The crucial point to note about the automorphism σ_λ is that (a) it ensures that the field Φ and the rescaled field $S_\lambda\Phi$ live in the same algebra (so that they may be compared), and that (b) it is constructed in such a way that the rescaled field $S_\lambda\Phi$ is again local in the sense of Def. 3.2. The factor λ^{4n} has been included in the definition of the scaling map S_λ in order to compensate for the fact that the quantum fields are distributions and therefore transform as densities under rescalings of the metric. The action of S_λ on some simple local, covariant fields is given below.

Next, we introduce the notion of the scaling dimension of a local, covariant field.

Definition 4.1. The scaling dimension d_Φ of a local, covariant field Φ is defined by

$$d_\Phi = \inf \{ \delta \in \mathbb{R} \mid \lim_{\lambda \rightarrow 0+} \lambda^{-\delta} S_\lambda \Phi = 0 \}, \quad (49)$$

where the limit is understood to mean that

$$\lim_{\lambda \rightarrow 0+} \lambda^{-\delta} S_\lambda \Phi[\mathbf{g}, p](f) = 0$$

for all metrics \mathbf{g} , all values of the parameters p and all test functions f .

It is easy to see from the definition that the free field indeed scales as $S_\lambda\varphi = \lambda\varphi$. The local c-number field $C = m^2 R 1$ scales as $S_\lambda C = \lambda^4 C$, so it has scaling dimension four. The fields in the above examples scale homogeneously. However, this is clearly not always so, as may be seen from the elementary example $(1 + R^2)^{-1} 1$, which is local, has scaling dimension zero, but which does not scale homogeneously (and which also has no well-defined engineering dimension).

We would like to require that our local Wick powers and local time ordered products scale homogeneously, the basic idea being that we wish our fields to have a well-defined engineering dimension. However, as it is well known in quantum field theory—and, as we shall see in more detail for the local Wick products below—logarithmic terms cannot be avoided in general (with the exception of the free field). Consequently, we will require, instead, that the local Wick powers and their local time ordered products scale “homogeneously up to logarithmic terms”. This requirement is formulated precisely as follows. We say that an element $a \in \mathcal{W}_p(M, \mathbf{g})$ has *order* k if its $(k + 1)$ times repeated commutator with a free field vanishes. (Prop. 2.1 provides a characterization of such elements.) By the expansion requirement, we know that the time ordered products $T(\varphi^{k_1} \dots \varphi^{k_n})$ have order $\sum_i k_i$. It is also clear that the order is additive under the multiplication of two operators. Using the notion of the order of an operator, we now give a recursive definition of local, covariant field with “almost homogeneous scaling”.

Definition 4.2. A local, covariant field Φ of order zero (i.e., a local c-number field) is said to have “almost homogeneous scaling” if it scales in fact exactly homogeneously,

$$\lambda^{-d_\Phi} S_\lambda \Phi = \Phi. \quad (50)$$

A local, covariant field Φ of order $k > 0$ is said to scale almost homogeneously if

$$\lambda^{-d_\Phi} S_\lambda \Phi = \Phi + \sum_i \ln^i \lambda \cdot \Psi_i, \quad \text{for all } \lambda > 0, \quad (51)$$

where the Ψ_i are finitely many local, covariant fields of order $\leq k - 1$ with $d_{\Psi_i} = d_\Phi$ and almost homogeneous scaling.

Our requirement concerning the scaling of local Wick-products and time ordered products is then the following.

Scaling: The local time ordered products $\Phi = T(\varphi^{k_1} \dots \varphi^{k_n})$ have almost homogeneous scaling with $d_\Phi = \sum k_i = \text{order of } \Phi$.

5 Analysis of the renormalization ambiguity for local Wick products and their time ordered products

5.1 Uniqueness of local Wick products

We now analyze the ambiguity in defining local Wick powers with the properties stated in the previous section. As previously mentioned, we will explicitly consider only undifferentiated Wick powers here, but our results can be straightforwardly extended to differentiated Wick powers (modulo the remark in section 4.1 above).

Theorem 5.1. Suppose we are given two sets of local Wick products $\varphi^k(x)$ and $\tilde{\varphi}^k(x)$, satisfying the requirements formulated in the previous section (for all k). Then there holds

$$\tilde{\varphi}^k(x) = \varphi^k(x) + \sum_{i=0}^{k-2} \binom{k}{i} C_{k-i}(x) \varphi^i(x). \quad (52)$$

Here,

$$C_k(x) \equiv C_k[g_{ab}(x), R_{abcd}(x), \dots, \nabla_{e_1} \dots \nabla_{e_{k-2}} R_{abcd}(x), \xi, m^2] \quad (k \in \mathbb{N}) \quad (53)$$

are polynomials (with real coefficients depending analytically on ξ) in the metric, the curvature and the mass parameter, which scale as $C_k \rightarrow \lambda^k C_k$ under rescalings $g_{ab} \rightarrow \lambda^{-2} g_{ab}$, $m^2 \rightarrow \lambda^2 m^2$, $\xi \rightarrow \xi$.

Remark: The space of possible curvature terms C_k described in the theorem is finite dimensional for every k . For example C_2 must be a real linear combination of R and m^2 , since these are the only curvature terms with the required properties. Therefore the ambiguity in defining φ^2 is given by $\tilde{\varphi}^2 = \varphi^2 + (Z_1 R + Z_2 m^2)1$, where Z_1, Z_2 are undetermined real constants, depending analytically on ξ .

Proof of Thm. 5.1: The proof is divided into two steps: We first show that there exist local, covariant, Hermitian c-number fields C_k such that Eq. (52) holds and which have the property that each C_k depends continuously (analytically) on the metric and scales homogeneously up to logarithmic terms with dimension $d_{C_k} = k$. The second step is then to show that the C_k are polynomials in the metric, the Riemann tensor, its derivatives and the coupling constants, and that they scale in fact exactly as $C_k \rightarrow \lambda^k C_k$ under a rescaling of the metric and the mass parameter.

The first step is accomplished by a simple induction argument in k . Clearly, Eq. (52) holds for $k = 1$ and $C_1 = 0$, since there is no ambiguity in the definition of the free field. Suppose we have found Hermitian local c-number fields C_i , $i = 2, 3, \dots, k-1$ such that Eq. (52) holds up to order $k-1$ and which have furthermore the properties (a) they are continuous (analytic) under corresponding variations of the metric and the parameters and (b) they have almost homogeneous scaling with dimension $d_{C_i} = i$. We define a local, covariant field Φ_k by

$$\Phi_k(x) \stackrel{\text{def}}{=} \tilde{\varphi}^k(x) - \left(\varphi^k(x) + \sum_{i=1}^{k-2} \binom{k}{i} C_{k-i}(x) \varphi^i(x) \right). \quad (54)$$

By the induction assumption it follows that the local, covariant field Φ_k is Hermitian, it is continuous (analytic) under corresponding variations of the metric and the parameters, and it has almost homogeneous scaling with $d_{\Phi_k} = k$. This is because Φ_k arises as a sum of local, covariant fields with these properties. Using the expansion requirement for the local Wick powers and the inductive assumption, one easily gets

$$[\Phi_k(x), \varphi(y)] = 0 \quad \text{for all } x, y \in M. \quad (55)$$

Using Prop. 2.1 we therefore get that $\Phi_k = C_k 1$, where $C_k \equiv C_k[\mathbf{g}, p]$ is some Hermitian local, covariant c-number field with the properties (a) and (b). Using the microlocal spectrum condition for the local Wick monomials, we moreover immediately get that C_k is actually a smooth function in x . We have thus completed the first step and we come to the second step.

The locality requirement, Def. 3.1, implies that⁵

$$\chi^* C_k[\mathbf{g}, p] = C_k[\chi^* \mathbf{g}, p], \quad (56)$$

⁵Note that the role played by ι_χ in the locality requirement is trivial in the case at hand, since C_k is a c-number.

for any diffeomorphism χ of M , and that $C_k[\mathbf{g}, p](x) = C_k[\mathbf{g}', p](x)$ holds true whenever $\mathbf{g} = \mathbf{g}'$ in some open neighborhood of the point x . The first condition means that $C_k[\mathbf{g}, p](x)$ is given by a diffeomorphism covariant expression, and the second means that it depends only on the germ of \mathbf{g} at x . In order to proceed, we now consider the subspace of all metrics \mathbf{g} , which are real analytic in some neighborhood of x , and we view C_k as a functional on that sub-space. Since the germ at x of a real analytic metric \mathbf{g} depends only on the metric itself and all its derivatives at x , this functional must be of the form

$$C_k[\mathbf{g}, p](x) \equiv C_k[g_{\mu\nu}(x), \overset{\circ}{\partial}_\sigma g_{\mu\nu}(x), \overset{\circ}{\partial}_\sigma \overset{\circ}{\partial}_\rho g_{\mu\nu}(x), \dots, p] \quad (57)$$

for all real analytic metrics \mathbf{g} . Here, $\overset{\circ}{\partial}_\mu$ is the coordinate derivative operator in some fixed analytic coordinate system around x and greek indices denote the components in these coordinates. For convenience, we take the values of all the coordinates of x to be zero. Consider, now, the 1-parameter family of coupling parameters $p^{(s)} = (\xi, s^2 m^2)$ and the following 1-parameter family of real analytic metrics, defined by

$$\mathbf{g}^{(s)} = s^{-2} \chi_s^* \mathbf{g}. \quad (58)$$

Here, χ_s is the diffeomorphism which in our coordinates around x acts by rescaling the coordinates by a factor s . Let y^α denote the coordinates of a point y in a sufficiently small neighborhood of x . In terms of components in our fixed coordinate system, we have

$$g_{\mu\nu}^{(s)}(y^\alpha) = g_{\mu\nu}(sy^\alpha). \quad (59)$$

It follows immediately from (59) that $\mathbf{g}^{(s)}$ is an analytic family of metrics in a neighborhood of x and $s = 0$. By the analyticity and analytic microlocal scaling degree requirements, $C_k[\mathbf{g}^{(s)}, p^{(s)}](x)$ is analytic in s in a neighborhood of $s = 0$, and we may thus expand it in a convergent power series about $s = 0$. It also follows immediately from (59) that $\overset{\circ}{\partial}_{\sigma_1} \dots \overset{\circ}{\partial}_{\sigma_k} g_{\mu\nu}^{(0)}(y) = 0$ for all y in a neighborhood x and that

$$g_{\mu\nu}^{(s)}(x) = g_{\mu\nu}(x) \quad \overset{\circ}{\partial}_{\sigma_1} \dots \overset{\circ}{\partial}_{\sigma_k} g_{\mu\nu}^{(s)}(x) = s^k \overset{\circ}{\partial}_{\sigma_1} \dots \overset{\circ}{\partial}_{\sigma_k} g_{\mu\nu}(x). \quad (60)$$

We find from this the power series expansion

$$C_k[\mathbf{g}^{(s)}, p^{(s)}](x) = \sum_{n=0}^{\infty} s^n \sum_{2j_0+j_1+2j_2+\dots+rj_r=n} \frac{\partial^{j_0+j_1+\dots+j_r} C_k[\dots]}{(\partial m^2)^{j_0} [\partial(\overset{\circ}{\partial} \mathbf{g}(x))]^{j_1} \dots [\partial(\overset{\circ}{\partial} \dots \overset{\circ}{\partial} \mathbf{g}(x))]^{j_r}} \times m^{2j_0} [(\overset{\circ}{\partial} \mathbf{g})(x)]^{j_1} \dots [(\overset{\circ}{\partial} \dots \overset{\circ}{\partial} \mathbf{g})(x)]^{j_r}, \quad (61)$$

where the spacetime indices have been omitted for simplicity and where

$$[\dots] = [g_{\mu\nu}(x), 0, \dots, 0, \xi, m^2 = 0].$$

Applying Eq. (56) to the diffeomorphism χ_s and using that $\chi_s(x) = x$, we get

$$C_k[\mathbf{g}^{(s)}, p^{(s)}](x) = C_k[s^{-2}\mathbf{g}, \xi, s^2 m^2](x). \quad (62)$$

Let us define $K_n[g_{\mu\nu}(x), \dots, \overset{\circ}{\partial}_{\sigma_1} \dots \overset{\circ}{\partial}_{\sigma_n} g_{\mu\nu}(x), \xi, m^2]$ (which we shall simply denote by $K_n[\mathbf{g}, \xi, m^2](x)$) as the coefficient of s^n in the above power series expansion,

$$C_k[s^{-2}\mathbf{g}, \xi, s^2 m^2](x) \equiv \sum_{n=0}^{\infty} s^n K_n[\mathbf{g}, \xi, m^2](x). \quad (63)$$

(Note that K_n is a polynomial in m^2 and the derivatives of the metric, whose coefficients depend analytically on ξ .) The left side of this identity is covariant under diffeomorphisms for all s . Therefore it follows that also each individual term in the series on the right side of this equation must have this property, i.e., for any analytic diffeomorphism χ ,

$$\chi^* K_n[\mathbf{g}, \xi, m^2] = K_n[\chi^* \mathbf{g}, \xi, m^2] \quad \text{for all } n \geq 0. \quad (64)$$

Since $K_n[\mathbf{g}, \xi, m^2](x)$ depends in addition polynomially on the metric and its derivatives at x , for all $x \in M$, it follows from the “Thomas replacement theorem” (see [13, Lem. 2.1]) that $K_n[\mathbf{g}, \xi, m^2]$ can be written in a “*manifestly covariant form*”, i.e., as a polynomial in the metric, the Riemann tensor, a finite number of its (symmetrized) metric derivatives and m^2 , whose coefficients depend analytically on ξ . In other words

$$K_n[\mathbf{g}, \xi, m^2](x) \equiv K_n[g_{ab}(x), R_{abcd}(x), \dots, \nabla_{(e_1} \dots \nabla_{e_{n-2})} R_{abcd}(x), \xi, m^2]. \quad (65)$$

We now use the scaling properties of C_k to find out more about its functional dependence on the metric and the coupling parameters. First, since the scaling dimension of C_k is k , we immediately find that $K_n = 0$ for all $n < k$. By Eq. (63), this means that the map $\lambda \rightarrow \lambda^{-k} C_k[\lambda^{-2}\mathbf{g}, \xi, \lambda^2 m^2](x)$ is analytic at $\lambda = 0$. Furthermore, we know that C_k is a local, covariant field which scales almost homogeneously. This means by definition that

$$\lambda^{-k} C_k[\lambda^{-2}\mathbf{g}, p(\lambda)] - C_k[\mathbf{g}, p] = \sum_i \ln^i \lambda \cdot \Psi_i[\mathbf{g}, p], \quad \text{with } p(\lambda) = (\xi, \lambda^2 m^2), \quad (66)$$

for a finite number of local, covariant fields Ψ_i . Since the left side of this equation is analytic at $\lambda = 0$ and since the logarithms are not, this is only possible if in fact $\Psi_i = 0$ for all i . Therefore, only the k -th term in the series (63) can be nonzero, which means that

$$C_k[\mathbf{g}, \xi, m^2](x) \equiv K_k[g_{ab}(x), R_{abcd}(x), \dots, \nabla_{(e_1} \dots \nabla_{e_{k-2})} R_{abcd}(x), \xi, m^2], \quad (67)$$

for all analytic metrics \mathbf{g} , that is, C_k is a polynomial in the metric, the curvature and the mass parameter, whose coefficients depend analytically on ξ . Since we already know

that C_k is Hermitian, the coefficients of this polynomial must be real. Moreover, we can directly read off from the expansion (63) that

$$C_k[\lambda^{-2}\mathbf{g}, \xi, \lambda^2 m^2](x) = \lambda^k C_k[\mathbf{g}, \xi, m^2]. \quad (68)$$

This then proves the theorem for analytic metrics \mathbf{g} . But we already know that $C_k[\mathbf{g}, p]$ has a continuous dependence on the metric. By approximating a smooth metric by a sequence of metrics which are real analytic in a neighborhood of x , we thus conclude that Eq. (67) must also hold for metrics which are only smooth, thus proving the theorem. \square

5.2 Existence of local Wick products

We next sketch how to construct local Wick powers with the desired properties. The construction is very similar to the construction for the renormalized stress energy operator given in [20]. The main ingredient in our construction is the local “Hadamard parametrix”, given by Eq. (4). H is not defined globally but only for x, y contained in a sufficiently small convex normal neighborhood.⁶ In the following we therefore restrict attention to such a neighborhood in all expressions involving H . (This does not create any problems for our construction of local Wick powers, since only coincident limits of quantities involving H need to be considered.) A technical complication arises from the fact that, while u is (at least locally) unambiguously defined for arbitrary smooth spacetimes, the same does not apply to v , which is unambiguously defined only for real analytic spacetimes. In the latter case, v is expandable as

$$v(x, y) = \sum_{n=0}^{\infty} v_n(x, y) \sigma^n, \quad (69)$$

where v_n are certain real and symmetric [17] smooth functions constructed from the metric and ξ, m^2 . In principle, one would like to define v by the above formula also for spacetimes which are only smooth. However, it is well-known that the above series does not in general converge in this case. This difficulty can be overcome by replacing the coefficients $v_n(x, y)$ in the above expansion by $v_n(x, y)\psi(\sigma/\alpha_n)$, where $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is some smooth function with $\psi(x) \equiv 1$ for $|x| < \frac{1}{2}$ and $\psi(x) \equiv 0$ for $|x| > 1$. If the α_n ’s tend to zero sufficiently fast, then the series with the above modified coefficients converges to a smooth function V . The coincidence limit of V and of all its derivatives does not depend on the choice of α_n and ψ , and it is only through these that V enters our definition of local Wick products. These choices therefore do not affect our definition.

We choose a quasi-free state ω on $\mathcal{A}(M, \mathbf{g})$ and represent $\mathcal{W}(M, \mathbf{g})$ as operators in the GNS representation of ω . Next, we define operator-valued distributions $:\varphi(x_1) \dots \varphi(x_n):_H$

⁶The reason for considering convex normal neighborhoods is that even σ is only defined for points that can be joined by a unique geodesic.

by a formula identical to Eq. (6), except that ω is replaced by H in that formula. Now, by the very definition of Hadamard states, H is equal, modulo a smooth function, to the symmetrized two-point function of ω . Consequently, it follows immediately that $:\varphi(x_1)\dots\varphi(x_n):_H$ can be smeared with distributions $t \in \mathcal{E}'_n(M, \mathbf{g})$ (supported sufficiently close to the total diagonal in M^n), and the so-obtained expressions belong to $\mathcal{W}(M, \mathbf{g})$. By analogy with our definition of a normal ordered field operator, Eq. (17), we are thus allowed to define

$$:\varphi^k(f):_H \stackrel{\text{def}}{=} \int_{M^k} :\varphi(x_1)\dots\varphi(x_k):_H f(x_1)\delta_{\mathbf{g}}(x_1, \dots, x_k) \prod_i \mu_{\mathbf{g}}(x_i). \quad (70)$$

Although it will not be needed until the next subsection, we find it convenient to define, by analogy with Eq. (19), also multi-local Wick products of the form $:\varphi^{k_1}(f_1)\dots\varphi^{k_n}(f_n):_H$. Local Wick products involving derivatives of the field can also be defined in a similar manner, although, as previously mentioned in the remark in Sec. 4.1, the definition fails to satisfy an additional condition that one may want to impose.

We claim that the fields $:\varphi^k:_H$ are local Wick monomials in the sense of the criteria given in Secs. 3 and 4. We will not give a detailed proof of this claim here but merely indicate the main arguments. That $:\varphi^k:_H$ is a local, covariant field immediately follows from the fact that the Hadamard parametrix is locally and covariantly defined in terms of the metric. The expansion property can be seen in just the same way as the corresponding property for normal ordered Wick monomials. It seems clear that the construction yields continuous (analytical) dependence of our Wick monomials under corresponding variations of the metric and the parameters, although we have not attempted to give a complete proof of this result. Finally, in order to verify the scaling axiom, we first restrict our attention to real analytic spacetimes (M, \mathbf{g}) , so that the function v , Eq. (69), is well-defined. In that case one finds from the definition of u and v that

$$\lambda^{-2}H[\lambda^{-2}\mathbf{g}, \xi, \lambda^2 m^2] = H[\mathbf{g}, \xi, m^2] + v[\mathbf{g}, \xi, m^2] \ln \lambda^2. \quad (71)$$

The appearance of the $v \ln \lambda^2$ term is due to the fact that the definition of H implicitly depends on a choice of length scale in the argument of the logarithm.⁷ Using Eq. (71) and the definition of the scaling map S_λ , Eq. (48), we find that $:\varphi^k:_H$ has dimension k and that it scales almost homogeneously in the sense of Def. 4.2. The same holds also for smooth spacetimes, by the continuity of the local Wick monomials. Thus we have demonstrated existence of local Wick products satisfying all of our requirements.

Although $:\varphi^k:_H$ scales almost homogeneously, it should be noted that the presence of the $\ln \lambda^2$ term in Eq. (71) implies that it fails to scale exactly homogeneously. The local, covariant fields Ψ_i in Eq. (51) are given by lower order local Wick monomials times

⁷This becomes more apparent by writing the logarithmic term in H as $v \ln \sigma \mu^2$, where μ has the dimension of a mass.

curvature terms of the appropriate dimension. Now, by Eq. (52), any other prescription for the local Wick products, φ^k , will be related to $:\varphi^k:_H$ by

$$\varphi^k(x) = :\varphi^k(x):_H + \sum_{i=0}^{k-2} \binom{k}{i} C_i(x) :\varphi^i(x):_H \quad (72)$$

where each C_i scales exactly homogeneously. It follows that φ^k also fails to scale exactly homogeneously. Consequently, by an argument given on pp. 98–99 of [20], there is an inherent ambiguity in the definition of φ^k that cannot be removed within the context of quantum field theory in curved spacetime. Thus, in quantum field theory in curved spacetime, the renormalization ambiguities arise not only from the definition of the time ordered products of Wick polynomials, but also from the local Wick polynomials themselves.

5.3 Uniqueness of local time ordered products

The analysis of the ambiguity in the definition of local time ordered products of local Wick monomials differs less in substance than in combinatorical complexity from the corresponding analysis for the local Wick products. Since the combinatorical side is rather well-known, we only sketch the proof of the result, Thm. 5.2. The presentation as well as the proof of our result is simplified by comparing an arbitrary prescription for the time ordered products to a prescription based on the local Wick products $:\varphi^k(x):_H$, defined in the previous subsection.

Again, for notational simplicity, we explicitly consider only time ordered products of undifferentiated local Wick products, but our arguments and results would apply to time ordered products of differentiated Wick products as well (modulo the remark of Sec. 4.1).

We find it convenient to use a multi-index notation, i.e. $k \in \mathbb{N}^n$ means a multi-index $k = (k_1, \dots, k_n)$, and standard abbreviations for multi-indices such as $\binom{k}{i} = \prod_j \frac{k_j!}{i_j!(k_j - i_j)!}$. $\mathcal{P} = I_1 \uplus \dots \uplus I_s$ denotes a collection of pairwise disjoint subsets of $\{1, \dots, n\}$.

Theorem 5.2. Consider a prescription $T(\prod_i :\varphi^{k_i}(x_i):_H)$ for defining local time ordered products based on the local Wick products $:\varphi^k(x):_H$, and another prescription, $\tilde{T}(\prod_i \varphi^{k_i}(x_i))$, based on another, arbitrary prescription $\varphi^k(x)$ for defining local Wick products. Assume that both prescriptions for defining local time ordered products satisfy all the requirements of Sec. 4. Then

$$\begin{aligned} \tilde{T}\left(\prod_{i=1}^n \varphi^{k_i}(x_i)\right) &= T\left(\prod_{i=1}^n :\varphi^{k_i}(x_i):_H\right) + \\ &+ \sum_{\substack{\mathcal{P} = I_1 \uplus \dots \uplus I_s \\ \text{not all } I_j = \emptyset}} T\left(\prod_{I=\{i_1, \dots, i_{|I|}\} \in \mathcal{P}} :\mathcal{O}_{k_I}(x_I):_H \prod_{i \notin I \forall I \in \mathcal{P}} :\varphi^{k_i}(x_i):_H\right), \quad (73) \end{aligned}$$

where $x_I = (x_{i_1}, \dots, x_{i_{|I|}})$ and $k_I = (k_{i_1}, \dots, k_{i_{|I|}})$. For $n \geq 2$, the $:\mathcal{O}_k(x_1, \dots, x_n):_H$ ($k \in \mathbb{N}^n$) are local, covariant quantum fields of the form

$$:\mathcal{O}_k(x_1, \dots, x_n):_H \equiv \sum_{i \leq k} \binom{k}{i} C_{k-i}(x_1) \delta(x_1, \dots, x_n) : \varphi^{i_1}(x_1) \dots \varphi^{i_n}(x_n) :_H \quad (74)$$

where the C_k are real c-number polynomials in $g_{ab}, R_{abcd}, \dots, \nabla_{(e_1} \dots \nabla_{e_{d-2})} R_{abcd}, m^2$, and covariant derivative operators $\nabla_a^{x_i}$, with scaling (= engineering) dimension $d = \sum k_i - 4(n-1)$, whose coefficients depend analytically on ξ . For $n = 1$, the quantum fields $:\mathcal{O}_k(x):_H$ ($k \in \mathbb{N}$) are given by the same kind of expression as above, but with no delta-functions and no covariant derivatives.

Remarks: (1) The multi-local covariant quantum fields $:\mathcal{O}_k(x_1, \dots, x_n):_H$ can alternatively be written as a sum of, possibly differentiated, mono-local Wick powers (i.e., depending only on one argument, say, the point x_1), multiplied by suitable differentiated delta-functions. In formulas, with (a) denoting a $4n$ -dimensional spacetime multi index,

$$:\mathcal{O}_k(x_1, \dots, x_n):_H = \sum_{(a)} :\mathcal{C}_k^{(a)}(x_1):_H \nabla_{a_1}^{x_1} \dots \nabla_{a_n}^{x_n} \delta(x_1, \dots, x_n), \quad (75)$$

where the $:\mathcal{C}_k^{(a)}:_H$ are local Wick polynomials, possibly with derivatives (all spacetime indices are assumed to be raised), whose coefficients are polynomials in the metric, the curvature, its covariant derivatives and the mass. These polynomials scale almost homogeneously with dimension $\sum_i k_i - 4(n-1)$.

The time ordered products appearing in the second line of Eq. (73) are to be understood as the expressions obtained by inserting the above expression for the fields $:\mathcal{O}_k:_H$ and by pulling the delta function type terms out of the time ordered product. The disadvantage of writing Eq. (73) explicitly in terms of these monolocal Wick-powers is that the relation between the ambiguities for different k and fixed order n (due to the expansion property of the time ordered products) now becomes a rather complicated-looking constraint on the possible delta-function type terms. A formulation of Thm. 5.2 not involving the specific prescription $:\varphi^k(x):_H$, but instead some other arbitrary prescription, would consist in writing all the generalized multilocal Wick products in expression (73) in terms of ordinary, monolocal ones, and then replacing these by that arbitrary prescription for those fields.

(2) The collection of local, covariant fields $:\mathcal{O}_k(x_1, \dots, x_i):_H$ with $i \leq n$ represent the finite renormalization ambiguity in defining time ordered products with n factors. The crucial point of the theorem is that the form of these ambiguities is severely restricted. Our uniqueness result for the Wick monomials, Thm. 5.1, is a special case of the above theorem, corresponding to $n = 1$.

Sketch of the proof for Thm. 5.2: One proceeds by a double induction in the order n in perturbation theory and the scaling dimension $d = \sum k_i$ of the time ordered products.

Assuming the validity of the theorem up to order $n - 1$, one finds, using the causal factorization of the time ordered products, that Eq. (73) also holds at order n , up to an unknown local, covariant $\Phi_k(x_1, \dots, x_n)$ which is nonzero only for points such that $x_1 = \dots = x_n$. Assuming now that this field has the form Eq. (74) for all multi indices k with $\sum k_i \leq d - 1$, one finds that it also has this form for dimension d , up to a c-number field of the form $c_k(x_1, \dots, x_n) = C_k(x_1)\delta(x_1, \dots, x_n)$, where C_k is a polynomial in the covariant derivative operators with bounded coefficients. By locality, C_k is locally constructed out of the metric and out of the coupling parameters. The task is then to show that it can be written as a polynomial in $g_{ab}, R_{abcd}, \dots, m^2, \nabla_a^{x_i}$, whose coefficients are analytic functions in ξ , and which scale as $C_k \rightarrow \lambda^d C_k$ under a corresponding rescaling of the parameters.

In order to find out more about the functional dependence of C_k on the metric, we now use the continuous and analytic dependence of the time ordered products under corresponding variations of the metric and the parameters, and their scaling behavior. This is done in essentially the same way as in our uniqueness proof for the local Wick products, so we only sketch the main arguments here, focusing on the differences compared to the case of the Wick monomials. For simplicity, let us first assume that C_k contains no derivatives. Consider an analytic family, $\mathbf{g}^{(s)}$, of analytic metrics in a neighborhood O in M , and an analytic family, $p^{(s)}$, of coupling parameters. We would like to show that the distribution $C_k^{(s)}(x)$ is analytic in s and x . (Here and in the following, the superscript s indicates that we mean the quantity associated with the metric $\mathbf{g}^{(s)}$ and the coupling parameters $p^{(s)}$.) In order to show this, we look at the analytic wave front set of $c_k^{(s)}(x_1, \dots, x_n)$, viewed as a distribution jointly in s and x_1, \dots, x_n . Now, this distribution arises as a sum of products of distributions of the form $c_j^{(s)}(x_1, \dots, x_m)$, with $m \leq n - 1$ and $j = (j_1, \dots, j_m)$, and of time ordered products, $T^{(s)}(\dots)$. The analytic wave front sets of the $c_j^{(s)}$ (viewed as distributions in s and the x -variables) is known by the inductive assumption; it has the same form as the wave front set of a delta-distribution. The analytic wave front set of the time ordered products—or rather of their expectation value in some analytic family of states, viewed as a distribution in s and the x -variables—is known by the analyticity requirement combined with the analytic microlocal spectrum condition. One can use this information to infer that $c_k^{(s)}(x_1, \dots, x_n)$ (viewed as a distribution in s and the x -variables) has analytic wave front set

$$\text{WF}_A(c_k) \subset \{(x_1, p_1, \dots, x_n, p_n, s, \rho) \in T^*(O^n \times (-\epsilon, \epsilon)) \setminus \{0\} \mid (x_1, p_1, \dots, x_n, p_n) \in \Gamma_A^T(O, \mathbf{g}^{(s)})\}, \quad (76)$$

where the conic set $\Gamma_A^T(O, \mathbf{g}^{(s)})$ is specified in the analytic microlocal spectrum condition. But we already know that c_k has support on the set of points such that $x_1 = \dots = x_n$.

Using this, we therefore find

$$\text{WF}_A(c_k) \subset \{(x_1, p_1, \dots, x_n, p_n, s, \rho) \in T^*(O^n \times (-\epsilon, \epsilon)) \setminus \{0\} \mid x_1 = \dots = x_n, \sum_i p_i = 0, \text{ not all } p_i = 0\}. \quad (77)$$

Now, we can trivially write

$$C_k^{(s)}(x) = \int_{M^{n-1}} c_k^{(s)}(x, y_1, \dots, y_{n-1}) f(y_1, \dots, y_{n-1}) \prod_{i=1}^{n-1} \mu^{(s)}(y_i), \quad (78)$$

where $f \in \mathcal{D}(O)$ is equal to one near x . By [11, Thm. 8.5.4'] we can conclude from this that that $C_k^{(s)}(x)$ —viewed as a distribution jointly in s and x —has analytic wave front set

$$\text{WF}_A(C_k) = \{(x, p, s, \rho) \mid (x, p, y_1, 0, \dots, y_{n-1}, 0, s, \rho) \in \text{WF}_A(c_k)\} = \emptyset$$

near x . Since x was arbitrary, this then shows that $C_k^{(s)}(x)$ is jointly analytic in x and s . We can now proceed as in the uniqueness proof for the local Wick products, by considering the particular family of metrics $\mathbf{g}^{(s)}$ (defined in (58)) and parameters $p^{(s)} = (\xi, s^2 m^2)$, and following through the same steps as there. This then shows us that C_k is indeed a polynomial in the metric, the curvature and the mass with engineering dimension d , whose coefficients depend analytically on ξ . The case when $C_k(x)$ also contains derivatives, $\nabla_a^{x_i}$, can be treated essentially in the same way as above. The only difference in the argument is that one has to consider more general functions f in Eq. (78). \square

An important direct consequence of Thm. 5.2 is the renormalizability of φ^4 -theory in curved spacetime, i.e., the perturbative quantum field theory corresponding to the classical theory given by the Lagrangian $\mathcal{L}_0 + \mathcal{L}_1$, where \mathcal{L}_0 is the free-field Lagrangian in Eq. (1), and where $\mathcal{L}_1 = f\varphi^4$. Observables in this interacting quantum field theory can be obtained from the S -matrix, given by

$$S(\mathcal{L}_1) = 1 + \sum_{n \geq 1} \frac{i^n}{n!} \int_{M^n} T(\mathcal{L}_1(x_1) \dots \mathcal{L}_1(x_n)) \mu_{\mathbf{g}}(x_1) \dots \mu_{\mathbf{g}}(x_n), \quad (79)$$

viewed here as a formal power series in the coupling constant f . We note that the above integrals would not in general make sense if f were taken to be a constant, so we instead take it to be an element in $\mathcal{D}(M)$ which is constant in some region, O , of spacetime, where we wish to define local observables. Choosing f in this way makes the series for $S(\mathcal{L}_1)$ truncated at some N an element in $\mathcal{W}(M, \mathbf{g})$.

Now $S(\mathcal{L}_1)$ clearly depends on what prescription for the local time ordered products one chooses in (79). So consider two different prescriptions, $T(\dots)$ and $\tilde{T}(\dots)$, for the time ordered products and denote the corresponding S -matrices by $S(\mathcal{L}_1)$ and $\tilde{S}(\mathcal{L}_1)$. Now if

it were true that $\tilde{S}(\mathcal{L}_1) = S(\mathcal{L}_1 + \delta\mathcal{L}_1)$ for some local, covariant field $\delta\mathcal{L}_1$ which had the same form as the original Lagrangian, then the theories based on different prescriptions for the time ordered products would actually be equivalent, the effect of $\delta\mathcal{L}_1$ being merely a redefinition of the coupling constants of the theory and of the field strength. Theories with this property are called “renormalizable”. It is well known that φ^4 -theory in Minkowski space belongs to this class of theories. We now show that Thm. 5.2 implies that this is also the case in curved spacetime.

Without loss of generality, we assume that one of the prescriptions for the time ordered products, say the “non-tilda” one, is based on local normal ordering prescription defined in the previous section. Since $:\mathcal{L}_1(x):_H = f(x):\varphi^4(x):_H$, we must investigate the possible form of the fields $:\mathcal{O}_k(x_1, \dots, x_n):_H$ in the case that all $k_i = 4$, because these govern the ambiguities in defining the time ordered products appearing in Eq. (79). Let us define a field $:\delta\mathcal{L}_1:_H$ by

$$\int_M :\delta\mathcal{L}_1(x):_H \mu_{\mathbf{g}}(x) \stackrel{\text{def}}{=} \sum_{n \geq 1} \int_{M^n} :\mathcal{O}_k(x_1, \dots, x_n):_H \prod_{i=1}^n f(x_i) \mu_{\mathbf{g}}(x_i), \quad (80)$$

where all $k_i = 4$, viewed as a formal power series in f . (When this series is truncated at some order N , the above equation defines a field in $\mathcal{W}(M, \mathbf{g})$.) It then follows from the properties of the fields $:\mathcal{O}_k:_H$ stated in Thm. 5.2 (applied to the case $k_i = 4$), and simple dimensional considerations that $:\delta\mathcal{L}_1:_H$ is given by

$$\begin{aligned} :\delta\mathcal{L}_1:_H = & \sum_{n \geq 1} f^n \left[Z_{0,n} :g^{ab} \nabla_a \varphi \nabla_b \varphi:_H + (Z_{1,n} R + Z_{2,n} m^2) : \varphi^2:_H + Z_{3,n} : \varphi^4:_H + \right. \\ & \left. (Z_{4,n} R^2 + Z_{5,n} R_{ab} R^{ab} + Z_{6,n} R_{abcd} R^{abcd} + Z_{7,n} \square R + Z_{8,n} m^2 R + Z_{9,n} m^4) 1 \right] + \dots, \quad (81) \end{aligned}$$

where “dots” denotes terms containing derivatives of f , and where $Z_{i,n}$ are real constants. One finds from Eq. (73), that

$$\tilde{S}(\mathcal{L}_1) = S(:\mathcal{L}_1:_H + :\delta\mathcal{L}_1:_H) \quad (82)$$

in the sense of formal power series of operators. Now $:\delta\mathcal{L}_1:_H$ has the same form as the original Lagrangian, $:\mathcal{L}_0:_H + :\mathcal{L}_1:_H$, apart from the terms proportional to the identity operator in the square brackets, and apart from the terms involving the derivatives of f . The terms proportional to the identity contribute only an overall phase to the S -matrix and therefore do not affect the definition of the interacting quantum fields derived from the S -matrix. The terms containing derivatives of f vanish in the formal limit when $f \rightarrow \text{const.}$, but for non-constant f they do affect the definition of the observables in the interacting theory. Nevertheless, it can be shown, using the arguments given in Sec. 8 of [2], that the interacting theory obtained from the interaction Lagrangian $:\mathcal{L}_1:_H + :\delta\mathcal{L}_1:_H$

locally (i.e., in the region O where f is constant) does not depend on the terms in $:\delta\mathcal{L}_1:_H$ involving derivatives of f .

This then proves renormalizability of φ^4 -theory in curved spacetime, provided of course that time ordered products satisfying our assumptions do indeed exist.

6 Conclusions and outlook

We have constructed, for every globally hyperbolic spacetime (M, \mathbf{g}) , an algebra $\mathcal{W}(M, \mathbf{g})$ containing normal ordered Wick products and time ordered products thereof. We then gave a notion of what it means for a field in that algebra to be “locally constructed out of the metric” in a covariant manner. Furthermore, we gave notions of analytic resp. continuous dependence of a local, covariant field under corresponding variations of the metric, and we gave a notion of “essentially homogeneous” scaling of a local, covariant field under suitable rescalings of the metric and the parameters of the theory. We then axiomatically characterized local Wick polynomials and local time ordered products by demanding that they satisfy the above requirements together with certain other, natural properties expected from a reasonable definition of these quantities. The imposition of these requirements was shown to reduce the ambiguities in defining these quantities to a finite number of real parameters. The nature of these ambiguities was shown to imply the renormalizability of a self-interacting quantum field theory in curved space. By an explicit construction, the existence of local Wick products with the desired properties was demonstrated. However, the issue of the existence of local time ordered products is beyond the scope of this paper and will be treated elsewhere.

We mention that our notion of the scaling of a local, covariant field makes possible a renormalization group analysis of the quantum observables in the interacting theory (posed as an open problem in [2]), i.e. an analysis of the behavior of an observable in the interacting theory under a change of scale. Namely, the “action of a renormalization group transformation” on an observable in the interacting theory is implemented in our framework by the scaling map, S_λ , defined in Eq. (48). The task is then to analyse the action of this map on observables in the interacting theory. Now, the observables in the interacting theory are defined in terms of perturbative expressions involving local time ordered products, and hence one only has to analyse the action of S_λ on the local time ordered products. Consider an expression of the form $T_\lambda(\dots) = \lambda^{-d} S_\lambda T(\dots)$, where $T(\dots)$ is a local time ordered product with scaling dimension d . The rescaled time ordered product $T_\lambda(\dots)$ is in general not equal to the unscaled time ordered product. However, by our uniqueness theorem 5.2, the scaled time ordered products differ from the unscaled ones by well-specified renormalization ambiguities, given by certain real parameters (depending on λ). As explained in the previous section, these parameters correspond to a finite renormalization of the coupling parameters in the theory. The action of S_λ (i.e., a renormalization group transformation) therefore translates directly

into a flow of the coupling parameters (and a multiplicative rescaling of the field strength). A detailed calculation of these can of course only be done based on a concrete prescription for the local time ordered products.

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7 Appendix

It is well known that the regularity properties of a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ are in correspondence with the decay properties of its Fourier transform. This can be made more precise by introducing the concept of the “wave front set” of a distribution [11], which we shall define now. Let u be a distribution of compact support. We define $\Sigma(u)$ to be the set of all $k \in \mathbb{R}^n \setminus \{0\}$ which have no conical⁸ neighborhood V such that

$$|\widehat{u}(p)| \leq C_N(1 + |p|)^{-N}$$

for all $p \in V$ and all $N = 1, 2, \dots$. $\Sigma(u)$ may be thought of as describing the “singular directions” of u . The wave front set provides a more detailed description of the singularities of a distribution by localizing these singular directions. If $u \in \mathcal{D}'(X)$, with X an open subset of \mathbb{R}^n , then we define $\Sigma_x(u) = \cap_f \Sigma(fu)$, where the intersection is taken over all $f \in \mathcal{D}(X)$ such that $f(x) \neq 0$. The wave front set of u is now defined as

$$\text{WF}(u) \stackrel{\text{def}}{=} \{(x, k) \in X \times (\mathbb{R}^n \setminus \{0\}) \mid k \in \Sigma_x(u)\}.$$

If $(x, k) \in \text{WF}(u)$, then x is a singular point of u , i.e., there is no neighborhood of x in which u can be written as a smooth function. Conversely, if x is a point such that no $(x, k) \in \text{WF}(u)$, then x is a regular point. Differentiation does not increase the wave front set, $\text{WF}(\partial u) \subset \text{WF}(u)$. The wave front set of a distribution is an entirely local concept, and it can be shown to transform covariantly under a change of coordinates, in the sense that $\text{WF}(\chi^*u) = (d\chi)^t \circ \text{WF}(u)$ for any diffeomorphism χ . This makes it possible to define in an invariant way the wave front set of distributions u on a manifold X . The above transformation property then shows that $\text{WF}(u)$ is intrinsically a (conic) subset of $T^*X \setminus \{0\}$, where T^*X denotes the cotangent bundle of X , and where $\{0\}$ means the zero section in T^*X . (In this paper, X is typically a product manifold $M \times \dots \times M$.)

In this paper we often use the notion of the wave front set to ensure that the pointwise product of certain distributions exists, or, more generally, to ensure that certain linear

⁸A cone in \mathbb{R}^n is a subset V with the property that if $k \in V$, then also $\lambda k \in V$ for all $\lambda > 0$.

maps with distributional kernel have a well-defined action on certain distributions (cf. Thms. 8.2.10 and 8.2.13 of ref. [11]). The above operations with distributions are not continuous (even if they are well defined) in the usual distribution topology. However, they are continuous in the so-called “Hörmander pseudo topology”, which is defined as follows: Let Γ be a closed conic set⁹ in $\mathbb{R}^n \times \mathbb{R}^n$, and let $\mathcal{D}'_\Gamma(\mathbb{R}^n)$ be the set of all distributions u on \mathbb{R}^n with $\text{WF}(u) \subset \Gamma$. We say that a sequence $\{u_\alpha\} \subset \mathcal{D}'_\Gamma(\mathbb{R}^n)$ converges to u in the Hörmander pseudo topology if $u_\alpha \rightarrow u$ in the usual sense of distributions and if, for any open neighborhood $O \subset \mathbb{R}^n$ and any cone $V \subset \mathbb{R}^n$ such that $\Gamma_x \subset V \ \forall x \in O$ and any $f \in \mathcal{D}(O)$ there holds

$$\sup_{k \notin V} |(\widehat{fu_\alpha} - \widehat{fu})(k)|(1 + |k|)^N \rightarrow 0 \quad \forall N \in \mathbb{N}.$$

This notion can be generalized in an invariant manner to smooth manifolds X , where Γ is now a closed conic subset of T^*X .

References

- [1] R. Brunetti, K. Fredenhagen and M. Köhler: “The microlocal spectrum condition and Wick polynomials on curved spacetimes,” *Commun. Math. Phys.* **180**, 633-652 (1996)
- [2] R. Brunetti and K. Fredenhagen: “Microlocal Analysis and Interacting Quantum Field Theories: Renormalization on physical backgrounds,” *Commun. Math. Phys.* **208**, 623-661 (2000)
- [3] T. S. Bunch: “BPHZ renormalization of $\lambda\Phi^4$ field theory in curved space-times,” *Ann. of Phys.* **131**, 118 (1981)
- [4] T. S. Bunch, P. Panangaden and L. Parker: “On renormalization of $\lambda\Phi^4$ in curved space-time I,” *J. Phys. A: Math. Gen.* **13**, 901-918 (1980), “On renormalization of $\lambda\Phi^4$ in curved space-time II,” *J. Phys. A: Math. Gen.* **13**, 919-932 (1980)
- [5] J. Dimock: “Algebras of Local Observables on a Manifold,” *Commun. Math. Phys.* **77**, 219-228 (1980)
- [6] M. Dütsch and K. Fredenhagen: “Algebraic quantum field theory, perturbation theory, and the loop expansion,” [hep-th/0001129]; “Perturbative algebraic field theory, and deformation quantization,” [hep-th/0101079]

⁹By this we mean a set of the form $\Gamma = \{(x, k) \in U \times \mathbb{R}^n \mid k \in \Gamma_x\}$, where U is a closed set and where Γ_x is a closed cone in \mathbb{R}^n for all $x \in U$.

- [7] J. J. Duistermaat and L. Hörmander: “Fourier integral operators II,” *Acta Math.* **128**, 183-269 (1972)
- [8] H. Epstein and V. Glaser: “The role of locality in perturbation theory,” *Ann. Inst. H. Poincaré Sec. A* **XIX**, 211–295 (1973)
- [9] K. Fredenhagen, private communication at Oberwolfach meeting, September 2000.
- [10] P. R. Garabedian: “Partial Differential Equations,” Wiley, New York 1964
- [11] L. Hörmander: “The Analysis of Linear Partial Differential Operators I”, Springer-Verlag, Berlin 1985
- [12] S. Hollands: “Aspects of Quantum Field Theory in Curved Spacetimes,” PhD thesis, University of York, September 2000
- [13] V. Iyer and R. M. Wald: “A Comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes,” *Phys. Rev. D* **52** 4430 (1995) [gr-qc/9503052]
- [14] B. S. Kay and R. M. Wald: “Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate Killing horizon,” *Phys. Rep.* **207**, 49 (1991)
- [15] B. S. Kay: “Casimir Effect in Quantum Field Theory,” *Phys. Rev. D* **20**, 3052-3062 (1979)
- [16] B. S. Kay: ‘Application of linear hyperbolic PDE to linear quantum fields in curved spacetimes: especially black holes, time machines and a new semilocal vacuum concept’, In: Proceedings “Journées Équations aux dérivées partielles”, GDR 1151 (CNRS), Nantes 2000, available at: <http://www.math.sciences.univ-nantes.fr/edpa/2000/html> and [gr-qc/0103056]
- [17] V. Moretti: “Proof of the symmetry of the off-diagonal Hadamard/Seeley-deWitt’s coefficients in C^∞ Lorentzian manifolds by a local Wick rotation,” *Commun. Math. Phys.* **212** 165-189 (2000) [gr-qc/9908068]
- [18] M. J. Radzikowski: “Micro-Local Approach to the Hadamard condition in QFT on Curved Space-Time,” *Commun. Math. Phys.* **179**, 529-553 (1996)
- [19] W. Tichy and E. Flanagan: “How unique is the expected stress energy tensor of a massive scalar field?,” *Phys. Rev. D* **58**, 124007 (1998) [gr-qc/9807015]

- [20] R. M. Wald: “Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics,” The University of Chicago Press, Chicago 1994
- [21] R. M. Wald: “General Relativity,” The University of Chicago Press, Chicago 1984
- [22] R. M. Wald: “The Back Reaction Effect in Particle Creation in Curved Spacetime,” Commun. Math. Phys. **54**, 1-19 (1977)